Sorting Introduction

CSE 326
Data Structures
Lecture 3

Plan

- Look at three sorting algorithms in detail
  - › Insertion Sort
  - › Mergesort
  - › Quicksort

Sorting

- Input
  - › an array A of data records
  - › a key value in each data record
  - › a comparison function which imposes a consistent ordering on the keys
- Output
  - › reorganize the elements of A such that
    - For any i and j, if i < j then A[i] ≤ A[j]

Consistent Ordering

- The comparison function must provide a consistent *ordering* on the set of possible keys
  - › You can compare any two keys and get back an indication of a < b, a > b, or a = b (tricotomy)
  - › The comparison functions must be consistent
    - If `compare(a,b)` says a<b, then `compare(b,a)` must say b>a
    - If `compare(a,b)` says a=b, then `compare(b,a)` must say b=a
    - If `compare(a,b)` says a=b, then `equals(a,b) and equals(b,a)` must say a=b

Why Sort?

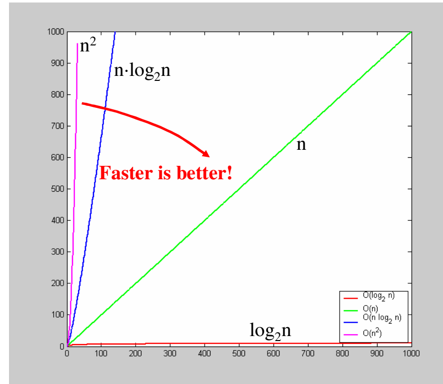
- Allows binary search of an N-element array in O(log N) time
- Allows O(1) time access to *k*th largest element in the array for any *k*
- Allows easy detection of any duplicates
- Sorting algorithms are among the most frequently used algorithms in computer science

Time

- How fast is the algorithm?
  - › The definition of a sorted array A says that for any i<j, A[i] ≤ A[j]
  - › This means that you need to at least check on each element at the very minimum
    - which is O(N)
  - › And you could end up checking each element against every other element
    - which is $O(N^2)$
  - › The big question is: How close to O(N) can you get?

Space

- How much space does the sorting algorithm require in order to sort the collection of items?
  - Is copying needed
  - In-place sorting – no copying – O(1) additional space.
  - External memory sorting – data so large that does not fit in memory



Insertion Sort

- What if first *k* elements of array are already sorted?
  - 4, 7, 12, 5, 19, 16
- We can shift the tail of the sorted elements list down and then *insert* next element into proper position and we get k+1 sorted elements
  - 4, 5, 7, 12, 19, 16

Insertion Sort Psuedocode

- Sort(array A of length N)
  - For i = 2…N
    - While A[i] > A[i-1] and i >= 0
      - Swap A[i] and A[i-1]
      - i = i -1

Insertion Sort

```
InsertionSort(A[1..N]: integer array, N: integer)
  {
  j, P, temp: integer ;
  for P = 2 to N {
    temp := A[P];
    j := P;
    while j > 1 and A[j-1] > temp do
          A[j] := A[j-1]; j := j-1;
    A[j]:= temp;
  }
}
```

- Is Insertion sort in-place?
- Running time = ?

Example

1	2	3	8	7	9	10	12	23	18	15	16	17	14
1	2	3	7	8	9	10	12	23	18	15	16	17	14
1	2	3	7	8	9	10	12	18	23	15	16	17	14
1	2	3	7	8	9	10	12	18	15	23	16	17	14
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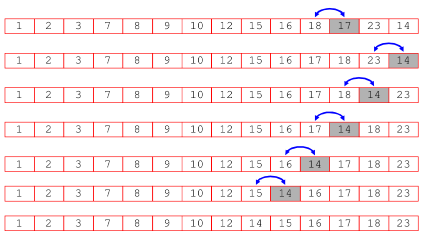
Example



Insertion Sort Characteristics

- In-place
- Running time
  - Worst case is $O(N^2)$
    - reverse order input
    - must copy every element every time
- Good sorting algorithm for almost sorted data
  - Each item is close to where it belongs in sorted order.

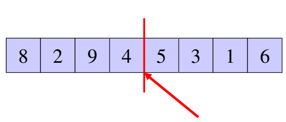
More Analysis

- Best Case?
  - O(N)
- Average Case? (assume no duplicates)
  - How many inversions?
    - How many unordered pairs? N(N+1)/2
    - Consider any A. Then pairs in A^R are inverted iff not inverted in A. So total number in A and A^R is N(N+1)/2. So average is N(N+1)/4.
  - Swap removes exactly one inversion, so Omega(N^2)

"Divide and Conquer"

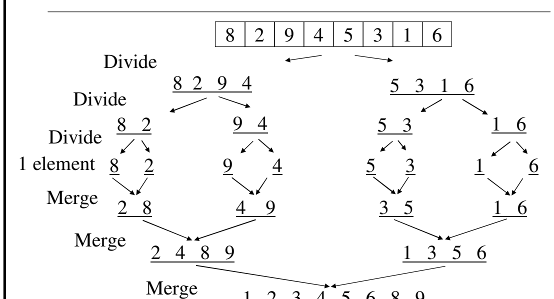
- Very important strategy in computer science:
  - Divide problem into smaller parts
  - Independently solve the parts
  - Combine these solutions to get overall solution
- **Idea 1**: Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves → known as Mergesort
- **Idea 2 :** Partition array into small items and large items, then recursively sort the two sets → known as Quicksort

Mergesort



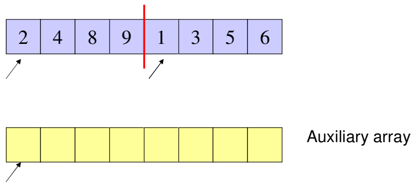
- Divide it in two at the midpoint
- Conquer each side in turn (by recursively sorting)
- Merge two halve together

Mergesort Example



Auxiliary Array

- The merging requires an auxiliary array.



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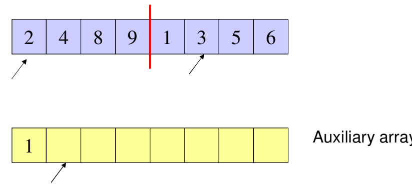

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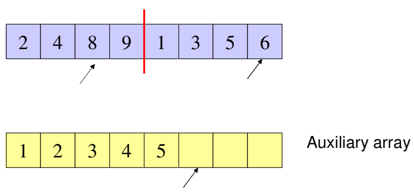
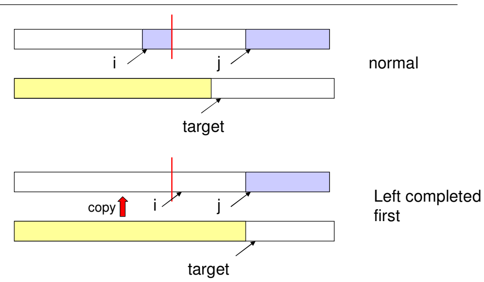
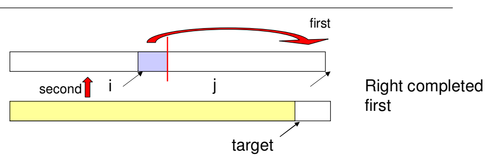


Merging



Merging



Merging

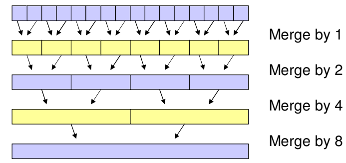
```
Merge(A[], T[] : integer array, left, right : integer) : {
  mid, i, j, k, l, target : integer;
  mid := (right + left)/2;
  i := left; j := mid + 1; target := left;
  while i ≤ mid and j ≤ right do
    if A[i] ≤ A[j] then T[target] := A[i] ; i:= i + 1;
      else T[target] := A[j]; j := j + 1;
    target := target + 1;
  if i > mid then //left completed//
    for k := left to target-1 do A[k] := T[k];
  if j > right then //right completed//
    k : = mid; l := right;
    while k ≥ i do A[l] := A[k]; k := k-1; l := l-1;
    for k := left to target-1 do A[k] := T[k];
}
```

Recursive Mergesort

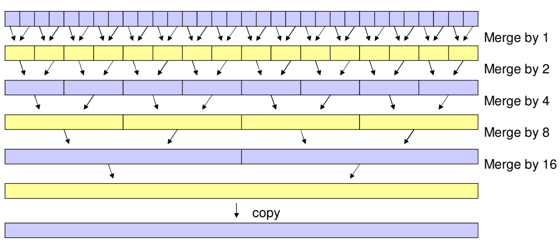
```
Mergesort(A[], T[] : integer array, left, right : integer) : {
  if left < right then
    mid := (left + right)/2;
    Mergesort(A,T,left,mid);
    Mergesort(A,T,mid+1,right);
    Merge(A,T,left,right);
}

MainMergesort(A[1..n]: integer array, n : integer) : {
  T[1..n]: integer array;
  Mergesort[A,T,1,n];
}
```


Iterative Mergesort



Iterative Mergesort



Iterative pseudocode

- Sort(array A of length N)
  - Let m = 2, let B be temp array of length N
  - While m<N
    - For i = 1…N in increments of m
      - merge A[i…i+m/2] and A[i+m/2…i+m] into B[i…i+m]
    - Swap role of A and B
    - m=m*2
  - If needed, copy B back to A

Mergesort Analysis

- Let T(N) be the running time for an array of N elements
- Mergesort divides array in half and calls itself on the two halves. After returning, it merges both halves using a temporary array
- Each recursive call takes T(N/2) and merging takes O(N)

Mergesort Recurrence Relation

- The recurrence relation for T(N) is:
  - $T(1) \le c$
    - base case: 1 element array → constant time
  - $T(N) \le 2T(N/2) + dN$
    - Sorting n elements takes
      - the time to sort the left half
      - plus the time to sort the right half
      - plus an O(N) time to merge the two halves
- T(N) = O(N log N)

Solving the Recurrence

$$\begin{aligned}T(n) &\le 2T(n/2)+dn \qquad \text{Assuming n is a power of 2}\\ &\le 2(2T(n/4)+dn/2)+dn\\ &= 4T(n/4)+2dn\\ &\le 4(2T(n/8)+dn/4)+2dn\\ &= 8T(n/8)+3dn\\ &\vdots\\ &\le 2^k T(n/2^k)+kdn\\ &= nT(1)+kdn \qquad \text{if } n=2^k\\ &\le cn+dn\log_2 n\\ &= O(n\log n)\end{aligned}$$

Properties of Mergesort

- Not in-place
  - Requires an auxiliary array
- Very few comparisons
- Iterative Mergesort reduces copying.

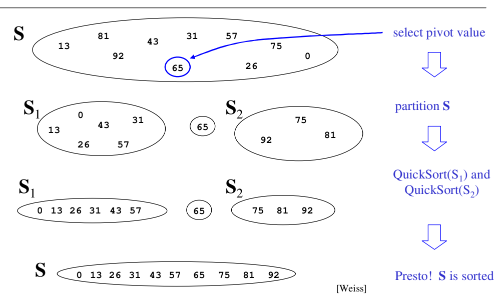
Quicksort

- Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does
  - Partition array into left and right sub-arrays
    - the elements in left sub-array are all less than pivot
    - elements in right sub-array are all greater than pivot
  - Recursively sort left and right sub-arrays
  - Concatenate left and right sub-arrays in O(1) time

"Four easy steps"

- To sort an array **S**
  - If the number of elements in **S** is 0 or 1, then return. The array is sorted.
  - Pick an element *v* in **S**. This is the *pivot* value.
  - Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ = {all values $x \le v$}, and $\mathbf{S}_2$ = {all values $x \ge v$}.
  - Return QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

The steps of QuickSort



Details, details

- "The algorithm so far lacks quite a few of the details"
- Implementing the actual partitioning
- Picking the pivot
  - want a value that will cause $|S_1|$ and $|S_2|$ to be non-zero, and close to equal in size if possible
- Dealing with cases where the element equals the pivot

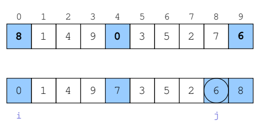
Quicksort Partitioning

- Need to partition the array into left and right sub-arrays
  - the elements in left sub-array are ≤ pivot
  - elements in right sub-array are ≥ pivot
- How do the elements get to the correct partition?
  - Choose an element from the array as the pivot
  - Make one pass through the rest of the array and swap as needed to put elements in partitions

Partitioning is done In-Place

- One implementation (there are others)
  - median3 finds pivot and sorts left, center, right
  - Swap pivot with next to last element
  - Set pointers i and j to start and end of array
  - Increment i until you hit element A[i] > pivot
  - Decrement j until you hit element A[j] < pivot
  - Swap A[i] and A[j]
  - Repeat until i and j cross
  - Swap pivot (= A[N-2]) with A[i]

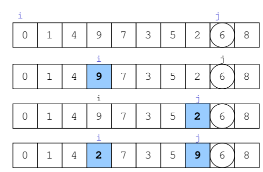
Example



Choose the pivot as the median of three.

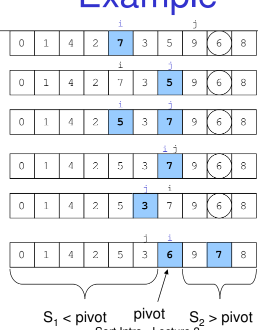
Place the pivot and the largest at the right and the smallest at the left

Example



Move i to the right to be larger than pivot.
Move j to the left to be smaller than pivot.
Swap

Example



Recursive Quicksort

```
Quicksort(A[]: integer array, left,right : integer): {
pivotindex : integer;
if left + CUTOFF ≤ right then
  pivot := median3(A,left,right);
  pivotindex := Partition(A,left,right-1,pivot);
  Quicksort(A, left, pivotindex - 1);
  Quicksort(A, pivotindex + 1, right);
else
  Insertionsort(A,left,right);
}
```

Don't use quicksort for small arrays.
CUTOFF = 10 is reasonable.

Alternative Pivot Rules

- Chose A[left]
  - Fast, but may be too biased
- Chose A[random], left $\leq$ random $\leq$ right
  - Completely unbiased
  - Will cause relatively even split, but slow
- Median of three, A[left], A[right], A[(left+right)/2]
  - The standard, tends to be unbiased, and does a little sorting on the side.

Quicksort Best Case Performance

- Algorithm always chooses best pivot and splits sub-arrays in half at each recursion
  - $T(0) = T(1) = O(1)$
    - constant time if 0 or 1 element
  - For $N > 1$, 2 recursive calls plus linear time for partitioning
  - $T(N) = 2T(N/2) + O(N)$
    - Same recurrence relation as Mergesort
  - $T(N) = O(N \log N)$

Quicksort Worst Case Performance

- Algorithm always chooses the worst pivot – one sub-array is empty at each recursion
  - $T(N) \leq a$ for $N \leq C$
  - $T(N) \leq T(N-1) + bN$
  - $\leq T(N-2) + b(N-1) + bN$
  - $\leq T(C) + b(C+1) + \ldots + bN$
  - $\leq a + b(C + C+1 + C+2 + \ldots + N)$
  - $T(N) = O(N^2)$
- Fortunately, *average case performance* is O(N log N) (see text for proof)

Properties of Quicksort

- No iterative version (without using a stack).
- Pure quicksort not good for small arrays.
- “In-place”, but uses auxiliary storage because of recursive calls.
- $O(n \log n)$ average case performance, but $O(n^2)$ worst case performance.

Folklore

- “Quicksort is the best in-memory sorting algorithm.”
- Truth
  - Quicksort uses very few comparisons on average.
  - Quicksort does have good performance in the memory hierarchy.
    - Small footprint
    - Good locality