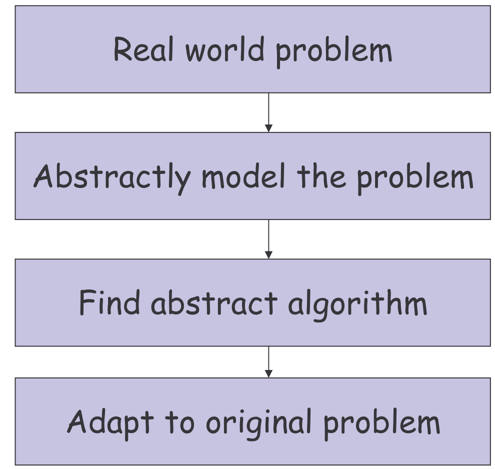
DFS, BFS, Shortest Path Problems

CSE 326
Data Structures
Unit 12

Reading: Sections 9.3, 9.6, 10.3.4

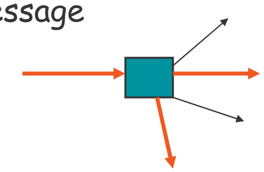
Applied Algorithm Scenario



Broadcasting in a Network

- Network of Routers
  - Organize the routers to efficiently broadcast messages to each other.

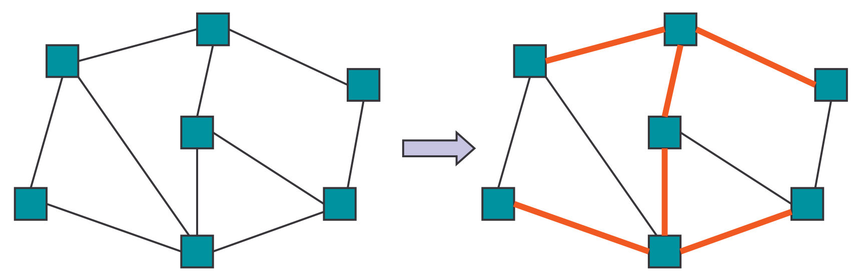
Incoming message



- Duplicate and send to some neighbors.
- Eventually all routers get the message

Goal: Minimize the number of messages.

Spanning Tree in a Graph



Vertex = router
Edge = link between routers

Spanning tree
- Connects all the vertices
- No cycles

Spanning Tree Problem

- Input: An undirected graph $G = (V,E)$. G is connected.
- Output: T contained in E such that
  - (V,T) is a connected graph
  - (V,T) has no cycles

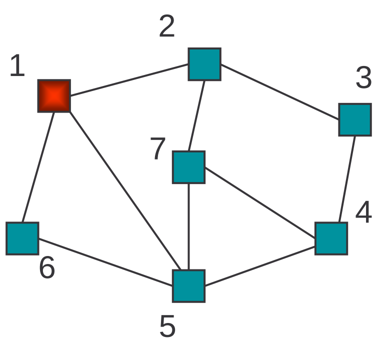
Depth First Search Algorithm

- Recursive marking algorithm
- Initially every vertex is unmarked

```
DFS(i: vertex)
  mark i;
  for each j adjacent to i do
    if j is unmarked then DFS(j)
end{DFS}
```

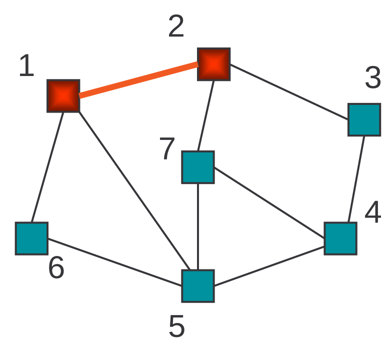

Example of Depth First Search

DFS(1)

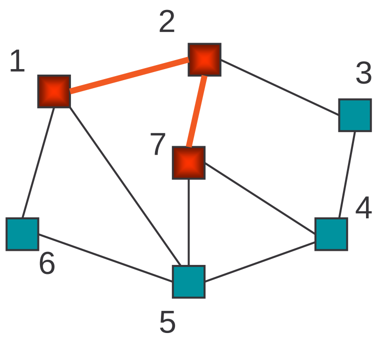


Example Step 2

DFS(1)
DFS(2)

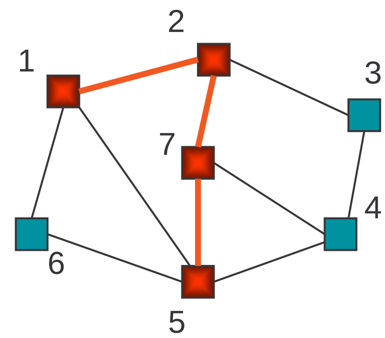


Example Step 3



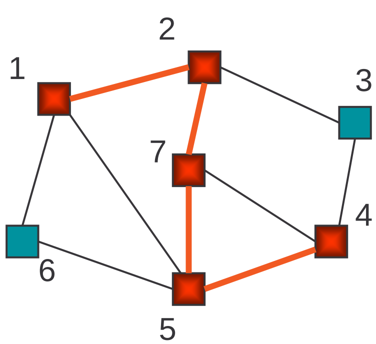
DFS(1)
DFS(2)
DFS(7)

Example Step 4



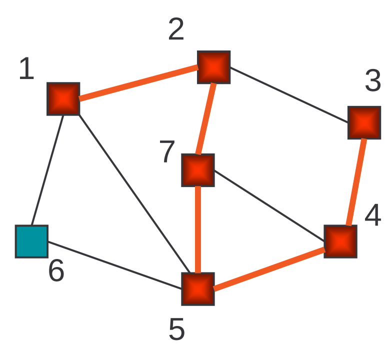
DFS(1)
DFS(2)
DFS(7)
DFS(5)

Example Step 5



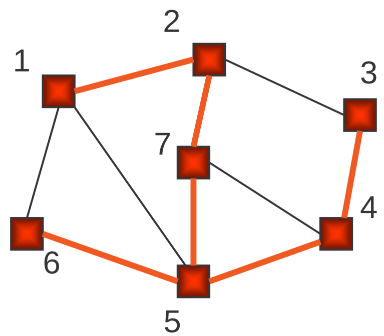
DFS(1)
DFS(2)
DFS(7)
DFS(5)
DFS(4)

Example Step 6



DFS(1)
DFS(2)
DFS(7)
DFS(5)
DFS(4)
DFS(3)

Example Step 7



DFS(1)
DFS(2)
DFS(7)
DFS(5)
DFS(4)
DFS(3)
DFS(6)

Note that the edges traversed in the depth first search form a spanning tree.

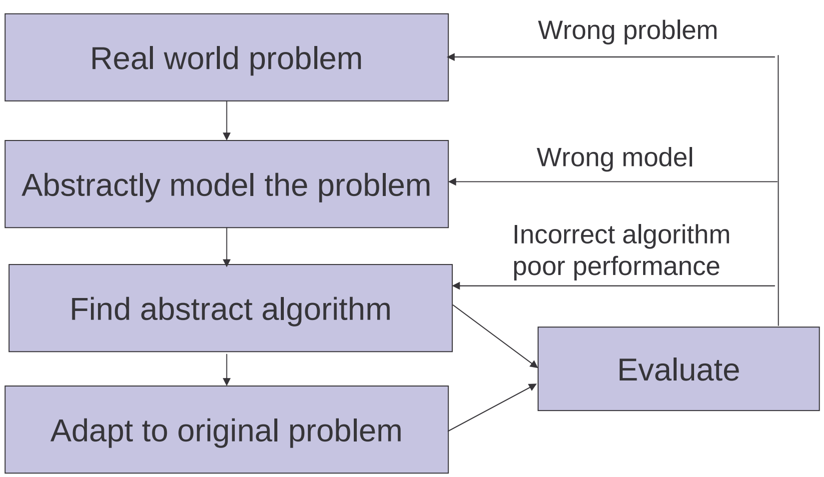
Spanning Tree Algorithm

```
Main
T := empty set;
ST(1);
end{Main}
```

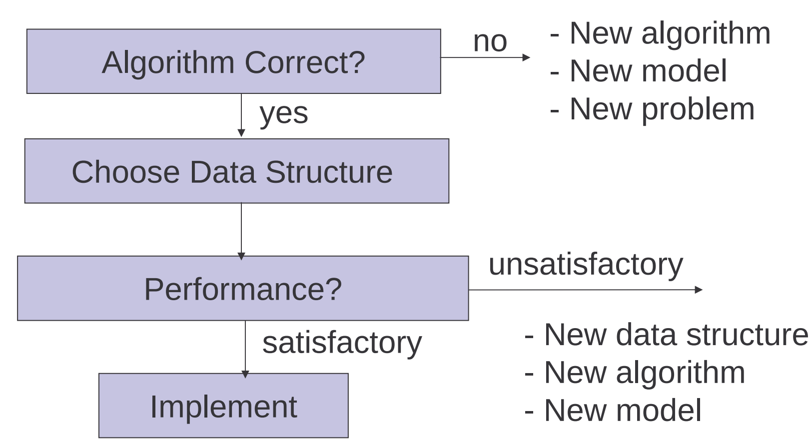
```
ST(i: vertex)
   mark i;
   for each j adjacent to i do
      if j is unmarked then
         Add {i,j} to T;
         ST(j);
end{ST}
```

The addition to DFS

Applied Algorithm Scenario

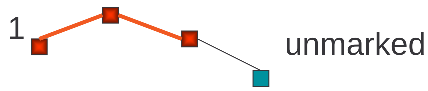


Evaluation Step Expanded



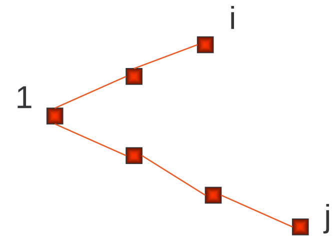
Correctness of ST Algorithm

- There are no cycles in T
  - This is an invariant of the algorithm.
  - Each edge added to T goes from a vertex in T to a vertex not in T.
- If G is connected then eventually every vertex is marked.

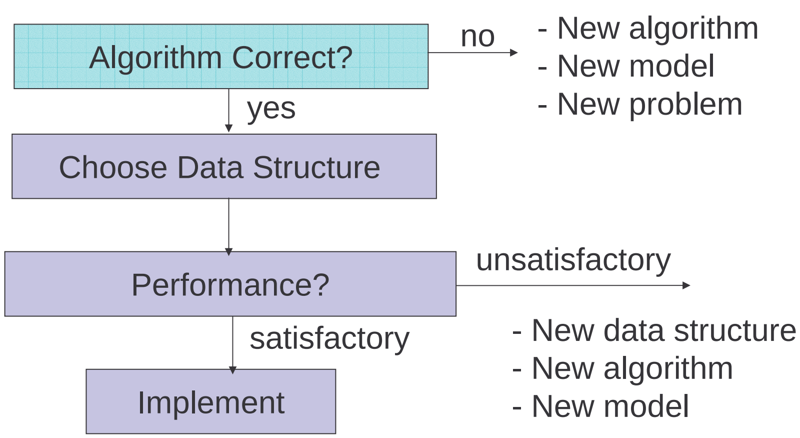


Correctness (cont.)

- If G is connected then so is (V,T)



Data Structure Step



Data Structure Choice

- Adjacency lists
  - Good for sparse graphs
  - Supports depth first search
- Adjacency matrix
  - Good for dense graphs
  - Supports depth first search

Spanning Tree with Adjacency Lists

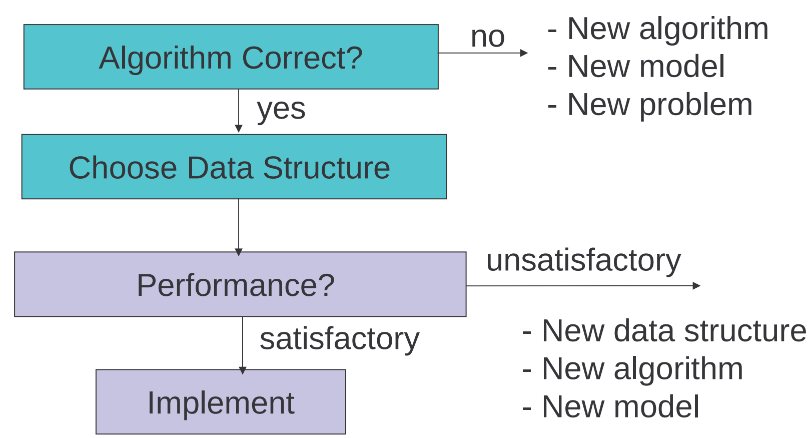
```
Main
   G is array of adjacency lists;
   M[i] := 0 for all i;
   T is empty;
   Spanning_Tree(1);
end{Main}
```

```
ST(i: vertex)
   M[i] := 1;
   v := G[i];
   while (v ≠ null)
      j := v.vertex;
      if (M[j] = 0) then
         add {i,j} to T;
         ST(j);
      v := v.next;
end{ST}
```

M is the marking array
(entry for each vertex).

Node of linked list:
vertex | | next

Performance Step



Performance of ST Algorithm

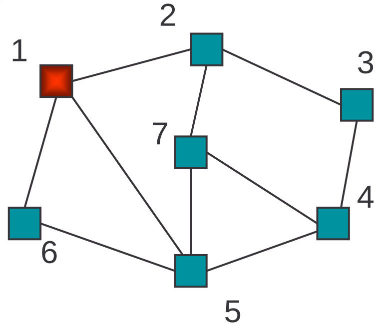
- n vertices and m edges
- Connected graph ($m \geq n-1$)
- Space complexity O(m)
- Time complexity O(m) - for each edge we perform O(1) operations in each of the two endpoints.

Other Uses of Depth First Search

- Popularized by Hopcroft and Tarjan 1973
- Connected components
- Strongly connected components in directed graphs
- Topological sorting of a acyclic directed graphs
- Maze solving

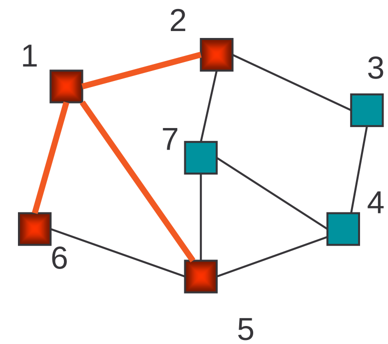
ST using Breadth First Search 1

- Uses a queue to order search



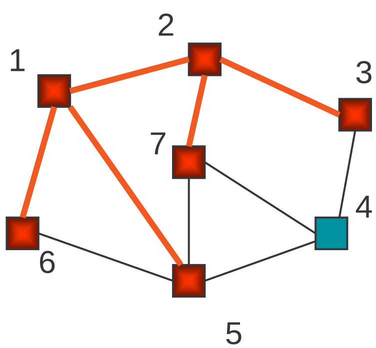
Queue = 1

Breadth First Search 2



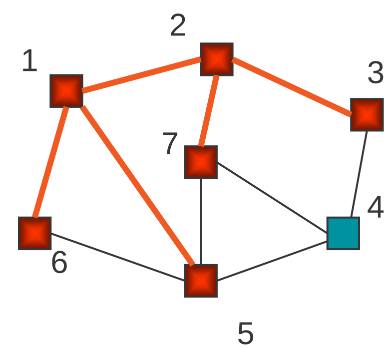
Queue = 2,6,5

Breadth First Search 3



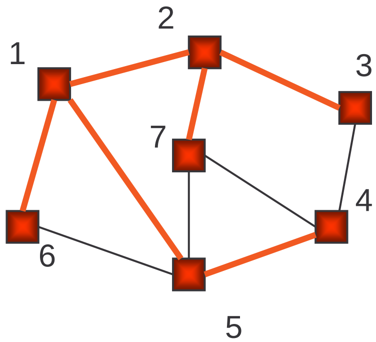
Queue = 6,5,7,3

Breadth First Search 4



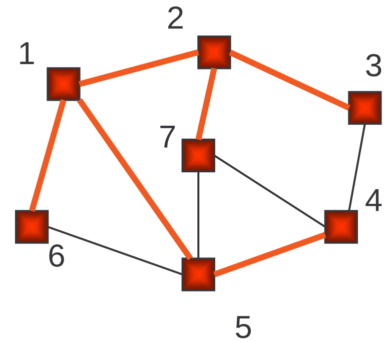
Queue = 5,7,3

Breadth First Search 5



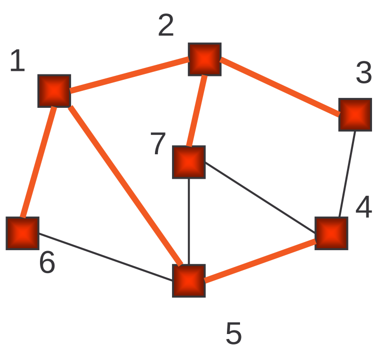
Queue = 7,3,4

Breadth First Search 6



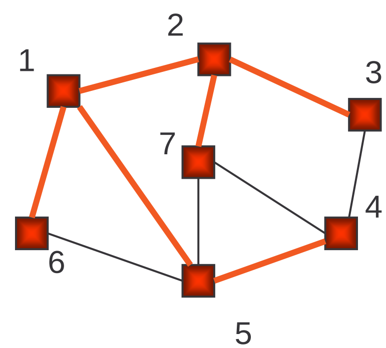
Queue = 3,4

Breadth First Search 7



Queue = 4

Breadth First Search 8



Queue =

Spanning Tree using Breadth First Search (BFS)

```
Initialize T to be empty;
Initialize Q to be empty;
Enqueue(1,Q) and mark 1;
while (Q is not empty) do
   i := Dequeue(Q);
   for each j adjacent to i do
      if j is not marked then
         add {i,j} to T;
         mark j;
         Enqueue(j,Q);
```


Depth First vs Breadth First

- Depth First
  - Stack or recursion
  - Many applications
- Breadth First
  - Queue (recursion no help)
  - Can be used to find shortest paths from the start vertex
- Both are O(|E|)

Shortest-path Algorithms

- Scenario: One router creates messages (source). Each message needs to reach other routers (one or more) along the shortest possible path.
- Abstraction: given a vertex s, find the shortest path from s to any other vertex of G.
- Other shortest path problems:
  - Different edges have different lengths (delay, cost, etc.)
  - All-pair shortest path problem: no specific source.

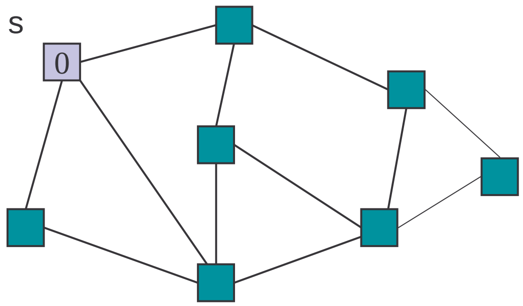
Using BFS for Shortest-path

- Given a vertex s, find the shortest path from s to any other vertex of G.

A 'centralized' version of BFS:

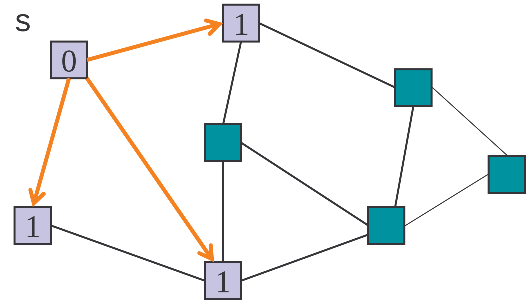
1. Label vertex s with 0.
2. $i \leftarrow 0$
3. Find all unlabeled vertices adjacent to at least one vertex labeled i. If none are found, stop.
4. Label all the vertices found in (3) with i + 1.
5. $i \leftarrow i + 1$ and go to (3).

BFS for Shortest Path (i=0)



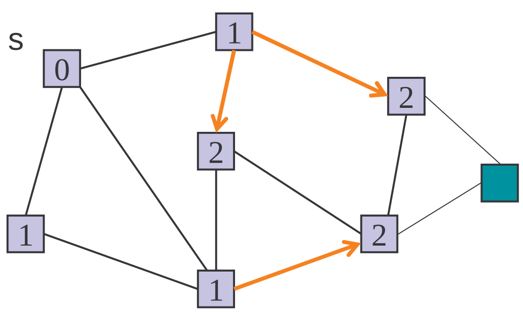
Vertices whose distance from s is 0 are labeled.

BFS for Shortest Path (i=1)



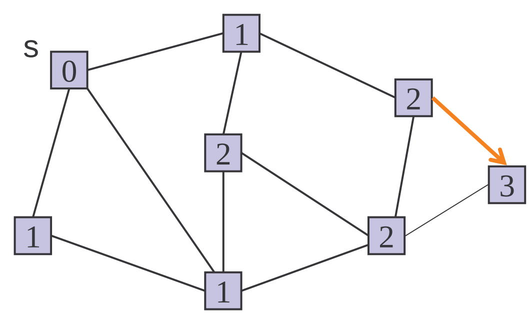
Vertices whose distance from s is 1 are labeled.

BFS for Shortest Path (i=2)



Vertices whose distance from s is 2 are labeled.

BFS for Shortest Path (i=3)



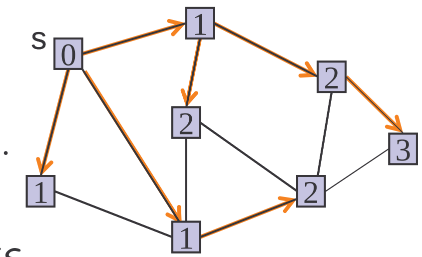
Vertices whose distance from s is 3 are labeled.

In the next iteration we find out that the whole graph is labeled and we stop.

The BFS Tree

Theorem: Each vertex is labeled by it its length from s.

Proof: By induction on the label.



For any v≠s, let p(v) be the vertex that 'discovered' v in BFS.

Then T={(p(v),v)} is a directed spanning tree rooted in s, and for each vertex v, the path from s to v in T is a shortest path from s to v in G.

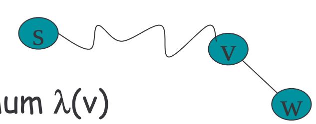
Note: the 'centralized' version is for simplification only. When implemented, we need the queue as before.

Single-Source Shortest Paths (Dijkstra's algorithm)

- Using BFS, we solve the problem of finding shortest path from s to any vertex v.
- What if edges have associated costs or distances? (BFS assumes edge costs are all 1.)
- Assume each edge (u,v) has non-negative weight c(u,v).
- A weight of a path = total weights of all edges on path.
- Problem: Find, for each vertex v, a shortest (minimum weight) path from s to v.

Idea of Dijkstra's Algorithm:

- Maintain:
  - λ[0..n-1] where λ(v) is the cost of best path from s to v found so far, and
  - T, set of vertices v for which λ(v) is not yet known to be optimal.
- Initially:
  - λ(s) = 0; λ(v) = ∞ for all v other than s.
  - T = V.
- In each step:
  - remove that v in T with minimum λ(v)
  - update those w in T s.t. (v,w) in E and λ(w) > λ(v) + c(v,w).



Dijkstra's Algorithm

Assumption: c(u,v) = ∞ if (u,v) not in E.

1. λ(s) ← 0 and for all *v ≠ s*,
λ(v) ← ∞.

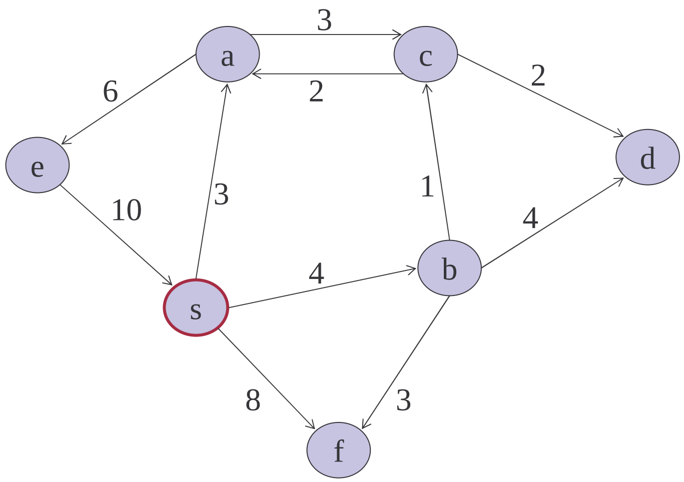
2. T ← V.

3. Let *u* be a vertex in *T* for which λ(u) is minimum.

4. For every edge, if *v∈T* and
λ(v) > λ(u) + c(u,v) then
λ(v) ← λ(u) + c(u,v).

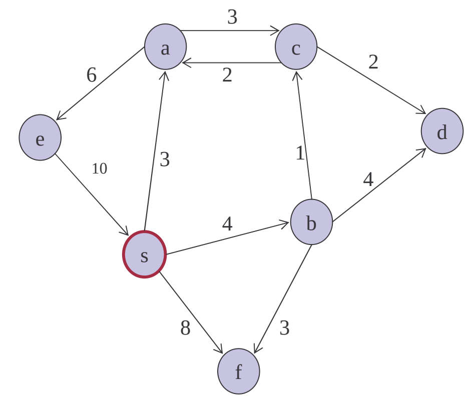
5. *T = T - {u}*, if T is not empty go to step 3.

Dijkstra's Algorithm - Example



Dijkstra's Algorithm - Example

	init	u=s	u=a
s	0	0 *	0 *
a	∞	3	3 *
b	∞	4	4
c	∞	∞	6
d	∞	∞	∞
e	∞	∞	9
f	∞	8	8

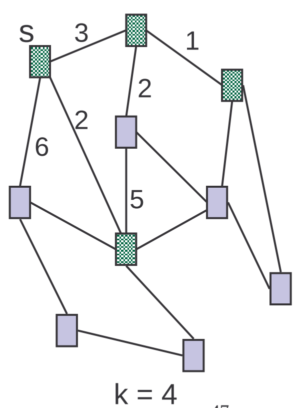


In class exercise: complete the execution.

* non-T vertices.

Why is this Algorithm Correct?

- **Theorem:** At the termination of the algorithm, λ(v) is the length of the shortest path from s to v for each vertex v of G.
- **Proof:** by induction on |V-T|.
- **Inductive hypothesis:** Let |V-T|=k.
  - ∀v in V-T, λ(v) is the length of the shortest path from s to v.
  - the vertices in V-T are the k closest vertices to s.
  - ∀v in T, λ(v) is the length of the shortest path from s to v that only goes through vertices in V-T.



Why is this Algorithm Correct?

- **Base case: |V-T|=1, T=V-{s}.**
  - for every v in V-T, λ(v) is the length of shortest path from s to v.
  - ✓ we init λ(s) =0.
  - the vertices in V-T are the k closest vertices to s.
  - ✓ V-T={s}. s is surely the closest to s.
  - for every v in T, λ(v) is the length of shortest path from s to v that only goes through vertices in V-T.
  - ✓ At this stage, λ(v) = ∞ for all v in V-T.

The λ values of vertices in V-T are correct and for each such v, the shortest path from s to v only goes through vertices in V-T

- Induction Step: Suppose true for first k steps. The SP to the $(k+1)^{st}$ closest vertex, say w, can go through only vertices in V-T, otherwise, there would be a closer vertex. Therefore, when selecting the min, we select the $(k+1)^{st}$ closest vertex to s.

Say w is added.

New λ value for a vertex x is min of old λ value and λ(w) + c(w,x)

Dijktra's Algorithm - Run Time Analysis

Implementation 1:

- Adjacency lists.
- An array for the λ values.

Complexity:

In each iteration:

1. Finding a vertex u in T with minimal λ
   In the whole execution: $n+(n-1)+(n-2)+\ldots+1 = O(n^2)$
2. Updating the λ-values of u's neighbors:
   In each iteration we check degree(u) values.
   The total sum of the degrees in 2m à O(m)

All together: $O(m+n^2)= O(n^2)$ (remember, $m \le n(n-1)$)

Dijktra's Algorithm - Run Time Analysis

- Implementation 2: data structure: priority queue
- Stores set S (in our case, this is T) such that there is a linear order on key values (in our case the key is the λ value).
- Supports operations:
  - Insert(x) - insert element with key value x into set.
  - FindMin() - return value of smallest element in set.
  - DeleteMin() - delete smallest element in set.
  - Find(x)

Priority-Queue Implementations

- Priority-Queue can be implemented such that each of these operations takes O(log n) time for sets of size n.

Running time of Dijkstra's algorithm:

We need to consider insertions, delete Mins, finds, modifying λ values.

Running Time of Dijkstra's Algorithm:

- n insertions: $O(n \log n)$ time
- n deleteMins: $O(n \log n)$ time
- m finds: $O(m \log n)$ time
- m λ-value modifications: $O(m \log n)$ time

- Running time: $O((n + m) \log n))$

- The $O(n^2)$ is better for dense graphs

Single-Source Shortest Paths (Bellman-Ford's algorithm)

- each edge (u,v) has a weight $c(u,v)$.
- $c(u,v)$ might be negative, but there are no negative cycles.

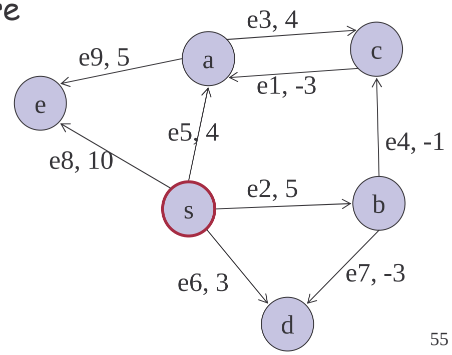
1. $\lambda(s) \leftarrow 0$ and for every $v \neq s$, $\lambda(v) \leftarrow \infty$.
2. As long as there is an edge such that $\lambda(v) > \lambda(u) + c(e)$ replace $\lambda(v)$ by $\lambda(u) + c(e)$.

For our purposes ∞ is not greater than $\infty + k$, even if k is negative.

Bellman-Ford algorithm

- How do we implement this algorithm?
- Order the edges: $e_1, e_2, \ldots, e_{|E|}$.
- Perform step 2 by first checking e_1, then e_2, etc., After the first such sweep, go through additional sweeps, until an entire sweep produces no improvement.

- Running Example:



BF algorithm – correctness and run time analysis

- Theorem: if a shortest path from s to v consists of k edges, then by the end of the k^{th} sweep v will have its final label.
- Proof: induction on k (not here).
- Since k is bounded by $|V|$ (remember, no negative cycles), step 2 is performed at most $|E| \cdot |V|$ times.
- Each comparison in step 2 can takes $O(1)$ if the graph is kept in an Adjacency Matrix (with the weights) and an array with the $\lambda(v)$ values.

à The time complexity of BF is $O(|E| \cdot |V|)$.

All-pair Shortest Path

- Input: a directed graph $G=(V,E)$ with $V = \{1, 2, \ldots, n\}$. The length of edge e is denoted by $c(e)$, and it may be negative.
- Output: All-pair shortest path: for any two vertices v,u in V, what is the shortest path from v to u
  - we will only be interested in the *length* of that path.

All-pair Shortest Path

- We can solve this problem using single-source shortest path algorithms. For example, we can run Bellman-Ford |V| times (one time for each possible selection of the source vertex s).
- Time complexity: $|V|*O(|V||E|)=O(|V|^2|E|)$
- We will see a solution using Dynamic Programming.

All-pair Shortest Path

Define

$$\delta^0(i,j) = \begin{cases} c(e) & \text{if } i \xrightarrow{e} j, \\ \infty & \text{if there is no edge from } i \text{ to } j. \end{cases}$$

Let $\delta^k(i,j)$ be the length of a shortest path from i to j among all paths which may pass through vertices $1,2,\ldots,k$ but do not pass through vertices $k+1, k+2,\ldots,n$.

Floyd Algorithm (1962)

1. Init $\delta^0(i, j)$ as defined earlier
2. $k \leftarrow 1$
3. For every $1 \le i, j \le n$ compute

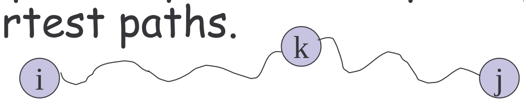
   $\delta^k(i, j) \leftarrow \text{Min} \{ \delta^{k-1}(i, j), \delta^{k-1}(i, k) + \delta^{k-1}(k, j)\}.$
4. If $k = n$, stop. If not, increment k and go to step 3.

Floyd Algorithm

$\delta^k(i, j) \leftarrow \text{Min} \{ \delta^{k-1}(i, j),$
$\delta^{k-1}(i, k) + \delta^{k-1}(k, j)\}.$

The shortest path from i to j which may pass through vertices $1,2,\ldots,k$ but do not pass through vertices $k+1, k+2,\ldots,n$:

1. Might not pass through vertex k, or
2. Might pass through k, and then it is composed by two already-computed shortest paths.



Floyd Algorithm

$\delta^k(i, j) \leftarrow \text{Min} \{ \delta^{k-1}(i, j),$
$\delta^{k-1}(i, k) + \delta^{k-1}(k, j)\}.$

Theorem: The value of $\delta^n(i, j)$ is the shortest path from i to j

Proof idea: By induction on k, the value of $\delta^k(i, j)$ is correct. In particular, for $\delta^n(i, j)$ we get the shortest path.

Floyd Algorithm

- The value of $\delta^n(i, j)$ is meaningful only if there are no negative cycles in G.
- The existence of negative cycles is detected by having $\delta^k(i, i) < 0$ for some i and k.
- Each application of step 3 requires n^2 operations, and step 3 is repeated n times. Thus, the algorithm is of complexity $O(n^3)$.

Shortest-path algorithms - Summary

- Single source, no weights:
  BFS - $O(m)$
- Single source, non-negative weights:
  Dijkstra $O((n + m) \log n))$ or $O(n^2)$
- Single source, arbitrary weights:
  Bellman-Ford: $O(nm)$
- All-pair shortest path, arbitrary weights: Floyd: $O(n^3)$