Lecture 26: El Grandé Finalé

- Agenda for the final class:
  - A Taste of Amortization (Chapter 11)
    - Aggregate method
    - Potential Method
    - Covered in Section 11.2 in the textbook
  - Final Review
    - Summary of what you've learned in this course

A Midterm Problem from Section BA/BB

- Consider the class **BigNum** with the method:
  `void addOne(); /* add one to the big number */`
- Implements a big number as a list of K binary digits
  - Each list cell contains a 0 or a 1
  - E.g. K = 10
  - E.g. 4 is represented as the list: 0000000100
  - E.g. 7 is represented as the list: 0000000111
- What does the run time for a given **`addOne()`** operation depend on?
- Run time depends on number of bits affected by the operation

A Midterm Problem from Section BA/BB

- Example:

  If the number is 0010101111

  After **`addOne()`**, number becomes: 0010110000

  Run time T = 5 because five bits were changed

- Question:

  Start with the number 0

  Consider a sequence of N **`addOne()`** operations

  What is the amortized running time as a function of N?

Tackling the **BigNum**: Naïve Strategy

- Question:

  Start with the number 0

  Consider a sequence of N **`addOne()`** operations

  What is the amortized running time as a function of N?

- Worst case:
  - K binary digits of which K-1 could be 1's
  - **`addOne()`** flips all K-1 bits and changes Kth bit to a 1
  - Worst case run time = K
  - For N operations, **Amortized run time = O(KN)**
  - Amortized run time per operation = O(KN)/N = O(K)

Is this a good bound?

Tackling the **BigNum**: A Better Strategy

- Amortized run time of O(Nk) is not a "tight" bound
  - Worst case occurs very rarely (not at all if N < k)
  - Worst case assumes every bit changes at every operation
- Can get a better bound by looking at how many times each bit can change during N **`addOne()`** operations

0000000000
0000000001
0000000010
0000000011
0000000100
0000000101
0000000110
0000000111
0000001000

0^{th} bit changes for every operation
1^{st} bit changes for every 2^{nd} operation
2^{nd} bit changes for every 4^{th} operation
3^{rd} bit changes for every 8^{th} operation
….

Every bit (change) counts

- For a sequence of N **`addOne()`** operations
  - 0^{th} bit changes N times
  - 1^{st} bit changes N/2 times
  - 2^{nd} bit changes N/4 times
  - 3^{rd} bit changes N/8 times
  - *i*th bit changes $N/2^i$ times
  - How big can *i* get for N **`addOne()`** operations?
    - It takes log N bits to represent the value N, max *i* = log N
- Total number of bit changes for N **`addOne()`** operations =

$$\sum_{i=0}^{\log N} N/2^i < N\sum_{i=0}^{\infty} 1/2^i = 2N$$

Amortized Analysis: The Aggregate Method

- **Amortized Run Time T(N)** for a sequence of N `addOne()` operations = O(total number of bit changes) = **O(N)**
  - Much better than the naïve bound on O(KN)
- **Amortized Run Time per operation** = O(N)/N = **O(1)**
  - Much better than the naïve bound of O(K)
- This is the aggregate method for amortized analysis
  - Basic Idea:
  - Calculate best possible big-oh bound T(N) on total run time of N operations (using a brute-force method)
  - Amortized run time per operation = T(N)/N

Amortized Analysis: The Potential Method

- Inspired by the concept of “potential energy” in physics
- Initial data structure D_0 on which N operations are performed
- We get D_i after applying ith operation to D_{i-1} at a cost of c_i
  - c_i is the run time of the ith operation
  - We do not know c_i but would like to put an upper bound on it
- Suppose we can come up with a “potential function” Φ that maps each D_i to a real number $\Phi(D_i)$
- Our goal: Use “potential function” Φ to put an upper bound on the total amortized cost of N operations

Amortized Analysis: The Potential Method

- Define the amortized cost of *i*th operation as:
$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

- Total amortized cost for N operations =
$$\sum_{i=1}^{N} \hat{c}_i = \sum_{i=1}^{N} \left(c_i + \Phi(D_i) - \Phi(D_{i-1})\right) = \sum_{i=1}^{N} c_i + \Phi(D_N) - \Phi(D_0)$$

What is the relation between this cost
and the total run time for N operations?

Amortized Analysis: The Potential Method

- Total amortized cost for N operations =
$$\sum_{i=1}^{N} \hat{c}_i = \sum_{i=1}^{N} \left(c_i + \Phi(D_i) - \Phi(D_{i-1})\right) = \sum_{i=1}^{N} c_i + \Phi(D_N) - \Phi(D_0)$$

- Then, if $\Phi(D_N) \geq \Phi(D_0)$ (or better, if $\Phi(D_i) \geq \Phi(D_0)$ for all i), then total run time for N operations =
$$T(N) = \sum_{i=1}^{N} c_i = O\left(\sum_{i=1}^{N} \hat{c}_i\right)$$

Back to **BigNum**: The Potential Method

- What should the "potential function" Φ be on the List D of binary digits?
  - Choice of Φ is somewhat of an art
  - Many choices may exist
  - But all should obey $\Phi(D_i) \geq \Phi(D_0)$ for all i
  - Some may give better bounds than others
- Think of what changes after each `addOne()` operation
- What about $\Phi(D_i) = n_i$ = number of 1's in the List after ith operation?
  - $\Phi(D_i) \geq \Phi(D_0)$ for all i
  - $\Phi(D_i)$ changes after every operation

BigNum: Grinding out its Potential

- $\Phi(D_i) = n_i$ = number of 1's in the List after ith operation
  - $\Phi(D_i) \geq \Phi(D_0)$ for all I
- Suppose ith operation changes x_i 1's to 0's
  - Cost c_i of ith operation = $x_i + 1$ (to change last 0 to 1)
- $\Phi(D_i)$ = Number of 1's after the ith operation = $n_{i-1} - x_i + 1$
- Amortized cost for ith operation =

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$
$$= (x_i + 1) + (n_{i-1} - x_i + 1) - n_{i-1} = 2$$

- Total amortized run time = $O\left(\sum_{i=1}^{N} \hat{c}_i\right)$ = O(2N) = **O(N)**

ART per operation O(1); Same as the aggregate method!

Other Applications

- Binomial Queues:
  - Starting from an empty queue, buildBinomialQueue takes O(N) rather than O(N log N) to insert N nodes
  - Analysis very similar to that for BigNum
  - Read Section 11.2 for the final
- Splay Trees
  - Result: Starting from an empty tree, M consecutive tree operations take O(M log N) time
  - Amortized run time per operation = O(log N)
  - Uses the potential function $\Phi(T)$ = sum over all nodes x in T of log(number of descendants of x)
  - Complicated analysis in Section 11.5 which you don't need to know for the final

Final Review

("We've covered way too much in this course…

What do I <u>really</u> need to know?")

Final Review: What you need to know

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

$$\sum_{i=0}^{N} A^i = \frac{A^{N+1}-1}{A-1}$$

- Basic Math
  - Logs, exponents, summation of series
  - Proof by induction
- Asymptotic Analysis
  - Big-oh, little-oh, Theta and Omega
  - Know the definitions and how to show f(N) is big-oh/little-oh/Theta/Omega of (g(N))
  - How to estimate Running Time of code fragments
    - E.g. nested "for" loops
- Recurrence Relations
  - Deriving recurrence relation for run time of a recursive function
  - Solving recurrence relations by expansion to get run time

What you need to know: Stacks, trees, …

- Lists, Stacks, Queues
  - Brush up on ADT operations – Insert/Delete, Push/Pop etc.
  - Array versus pointer implementations of each data structure
  - Header nodes, circular, doubly linked lists
- Trees
  - Definitions/Terminology: root, parent, child, height, depth etc.
  - Relationship between depth and size of tree
    - Depth can be between O(log N) and O(N) for N nodes

What you need to know: BSTs

- Binary Search Trees
  - How to do Find, Insert, Delete
    - Bad worst case performance – could take up to O(N) time
  - AVL trees
    - Balance factor is +1, 0, -1
    - Know single and double rotations to keep tree balanced
    - All operations are O(log N) worst case time
  - Splay trees – good amortized performance
    - A single operation may take O(N) time but in a sequence of operations, average time per operation is O(log N)
    - Every Find, Insert, Delete causes accessed node to be moved to the root
    - Know how to zig-zig, zig-zag, etc. to "bubble" node to top
  - B-trees: Know basic idea behind Insert/Delete

WYNTK: Priority Queues and Hashing

- Priority Queues
  - Binary Heaps: Insert/DeleteMin, Percolate up/down
    - Array implementation
    - BuildHeap takes only O(N) time (used in heapsort)
  - Binomial Queues: Forest of binomial trees with heap order
    - Merge is fast – O(log N) time
    - Insert and DeleteMin based on Merge
- Hashing
  - Hash functions based on the mod function
  - Collision resolution strategies
    - Chaining, Linear and Quadratic probing, Double Hashing
  - Load factor of a hash table

WYNTK: Sorting

- Sorting Algorithms: Know run times and how they work
  - Elementary sorting algorithms and their run time
    - Bubble sort, Selection sort, Insertion sort
  - Shellsort – based on several passes of Insertion sort
    - Increment Sequence
  - Heapsort – based on binary heaps (max-heaps)
    - BuildHeap and repeated DeleteMax's
  - Mergesort – recursive divide-and-conquer, uses extra array
  - Quicksort – recursive divide-and-conquer, Partition in-place
    - fastest in practice, but $O(N^2)$ worst case time
    - Pivot selection – median-of-three works best
  - Know which of these are stable and in-place
  - Lower bound on sorting, bucket sort, and radix sort

WYNTK: Disjoint Sets and Graphs

- Disjoint Sets and Union-Find
  - Up-trees and their array-based implementation
  - Know how Union-by-size and Path compression work
  - No need to know run time analysis – just know the result:
    - Sequence of M operations with Union-by-size and P.C. is $\Theta(M\ \alpha(M,N))$ – basically $\Theta(1)$ amortized time per op
- Graph Algorithms
  - Adjacency matrix versus adjacency list representation of graphs
  - Know how to Topological sort in $O(|V| + |E|)$ time using a queue
  - Breadth First Search (BFS) for unweighted shortest path

WYNTK: Graph Algorithms

- Graph Algorithms (cont.)
  - Dijkstra's shortest path algorithm – greed works!
    - Know how a priority queue can speed up the algorithm
  - Depth First Search (DFS)
  - Minimum Spanning Trees: Know the 2 greedy algorithms
    - Prim's algorithm – similar to Dijkstra's algorithm
    - Kruskal's algorithm
      - Know how it uses a priority queue and Union/Find
    - Euler versus Hamiltonian circuits – difference in run times
    - Know what P, NP, and NP-completeness mean
      - How one problem can be "reduced" to another (e.g. input to HC can be transformed into input for TSP)

WYNTK: Algorithm Design Techniques

- Greedy Algorithms
  - Bin Packing
- Divide & Conquer
  - Solving various types of recurrence relations for T(N)
- Dynamic Programming (Memoizing)
  - DP-Fibonacci
  - Go over other examples in text
- Randomized Data Structures and Algorithms
  - Average run time over all inputs vs. Expected run time for one input
  - Treaps
  - Primality Testing
- Backtracking and Game Trees

WYNTK: Amortized Analysis

- Know the basic concept
  - Amortized run time per operation over a sequence of N operations
- Two Techniques
  - Aggregate Method: Compute directly the total time for N operations and divide by N
  - Potential Method: Find a "potential function" that can be used to place an upper bound on total run time
  - E.g. Binary counter, binomial queues (see textbook)

WYNTK about the Final

- Details:
  - Covers Chapters 1-10, 11.2, 12.5 in the textbook
    - Emphasis on Chapters 7-10, Sec. 11.2, and 12.5
    - Emphasis on material covered in lecture slides
  - Closed book, closed notes except:
    - You may bring one 8 ½'' x 11'' sheet of notes
  - Time: 1 hour and 50 minutes
  - When: 8:30-10:20 a.m., Thursday, March 20 in class
  - Sample questions are on class website
  - Final will contain space for answers; no bluebooks
  - Bring pens/sharpened pencils (and sharpened minds!)
  - No, the final won't be NP-complete (it will be in P)

No class on Friday!

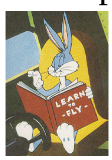
Final Exam:

Where: This room

When: 8:30-10:20 a.m., Thursday, March 20

To Do:

Go over practice final and problems on web site



Prepare, prepare, prepare…

N'joy da
spring break!