CSE 326: Data Structures
Topic 17: Becoming Famous with P and NP

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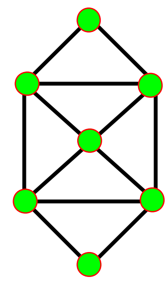
Euler Circuits

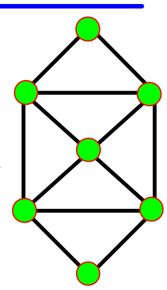


Can you traverse all edges exactly once, starting and finishing at the same vertex?

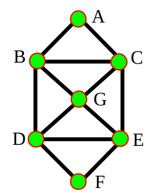
Possible if and only if:
1. Graph is connected
2. Each vertex has even degree

Finding Euler Circuits: DFS and then Splice

- Given a graph G = (*V*,*E*), find an Euler circuit in G
  - Can check if one exists in O(|*V*|) time **How?**
- Basic Euler Circuit Algorithm:
  1. Do a depth-first search (DFS) from a vertex until you are back at this vertex
  2. Pick a vertex on this path with an unused edge and repeat 1.
  3. Splice all these paths into an Euler circuit
- Running time =

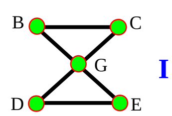


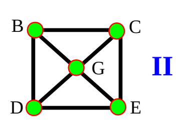
Euler Circuit Example



Euler with a Twist: Hamiltonian Circuits

- Euler circuit: A cycle that goes through each *edge* exactly once
- Hamiltonian circuit: A cycle that goes through each *vertex* exactly once
- Does graph **I** have:
  - An Euler circuit?
  - A Hamiltonian circuit?
- Does graph **II** have:
  - An Euler circuit?
  - A Hamiltonian circuit?





Finding Hamiltonian Circuits in Graphs

- Problem: Find a Hamiltonian circuit in a graph G = (*V*,*E*)
  - Sub-problem: Does G contain a Hamiltonian circuit?
  - Is there an easy (linear time) algorithm for checking this?
- Runtime?

Polynomial versus Exponential Time

- Most of our algorithms so far have been O(log N), O(N), O(N log N) or O(N^2) running time for inputs of size N
  - These are all *polynomial time* algorithms
  - Their running time is O(N^k) for some k > 0
- Exponential time B^N is asymptotically *worse than any* polynomial function N^k for any k
  - For any k, N^k is o(B^N) for any constant B > 1
- Polynomial time algorithms are generally regarded as "fast" algorithms – these are the kind we want!
- Exponential time algorithms are generally inefficient – avoid these!

The "complexity" class P

- The set P is defined as the set of all problems that can be solved in polynomial worse case time
  - Also known as the polynomial time complexity class – contains problems whose time complexity is O(N^k) for some k
- Examples of problems in P: searching, sorting, topological sort, single-source shortest path, Euler circuit, etc.

The "complexity" class NP

- Definition: NP is the set of all problems for which a given *candidate solution* can be *checked* in polynomial time
- Example of a problem in NP:
  - Our new friend, the Hamiltonian circuit problem: Why is it in NP?
- NP = "Non-Deterministic Polynomial Time"

Other problems in NP

- Sorting: Can test in linear time if a candidate ordering is sorted
- But sorting is also in P.
  - Are any other problems in P also in NP?

The Intimate Relationship between P and NP

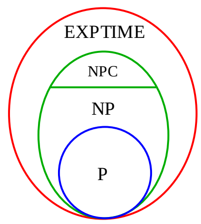
- Sorting is in P. Are any other problems in P also in NP?
  - YES!
  - All problems in P are also in NP i.e. P ? NP
  - If you can solve a problem in polynomial time, can definitely verify a solution in polynomial time
- So, some problems in NP like searching, sorting, etc. are also in P.
- Question: Are all problems in NP also in P?
  - Is NP ? P?

Your chance to win a Turing award: P = NP?

- Nobody knows whether NP ? P
  - Proving or disproving this will bring you instant fame!
- It is generally believed that P ? NP i.e. there are problems in NP that are not in P
  - But no one has been able to show even one such problem
- A very large number of problems are in NP (such as the Hamiltonian circuit problem) but not known to be in P
  - No one has found fast (polynomial time) algorithms for these problems
  - No one has been able to prove such algorithms don't exist (i.e. that these problems are not in P)!

P, NP, and Exponential Time Problems

- All algorithms for NP-complete problems so far have tended to run in nearly exponential worst case time
  - But this doesn't mean fast sub-exponential time algorithms don't exist! Not proven yet…
- Diagram depicts relationship between P, NP, and EXPTIME (class of problems that can be solved within exponential time)



It is believed that
P ? NP ? EXPTIME

NP-complete problems

- The "hardest" problems in NP are called NP-complete (NPC) problems
- Why "hardest"? A problem X is **NP-complete** if:
  1. X is in NP and
  2. *any* problem Y in NP can be *converted to* X in polynomial time such that solving X also provides a solution for Y

  (If only 2 holds, X is said to be **NP-hard**)

Input to Y ⟶ "Converter" Algorithm ⟶ Input to X
(runs in poly time)

We say that problem Y can be reduced to X

Note: X is NP-hard if all problems in NP can be reduced to X

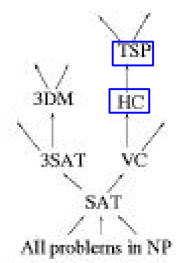
The Power of NP-completeness

All problems Y in NP ⟶ Convert input ⟶ Algorithm for X (any NPC problem) ⟶ Solution

- Thus, **if you find a poly time algorithm for just one NPC problem X, all problems in NP can be solved in poly time**
- Example: The Hamiltonian circuit problem can be shown to be NP-complete (not so easy to prove from scratch!)

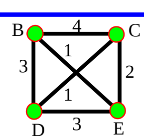
The "graph" of NP-completeness

- Cook first showed (in 1971) that satisfiability of Boolean formulas (SAT) is NP-complete
- Hundreds of other problems (from scheduling and databases to optimization theory) have since been shown to be NPC
- How? By giving an algorithm for **converting a known NPC problem to your pet problem in poly time**. Then, **your problem is also NPC**!



Showing NP-completeness: An Example

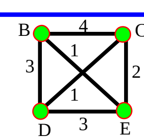
- Consider the **Traveling Salesperson (TSP) Problem**: Given a fully connected, weighted graph G = (V,E), is there a cycle that visits all vertices exactly once and has total cost ? K?
- TSP is in NP (why?)
- Can we show TSP is NP-complete? How?



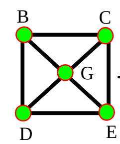
Cycle with cost ? 8 = BDCEB

Showing NP-completeness: An Example

- Can we show TSP is NP-complete?
  - We know Hamiltonian Circuit (HC) is NPC
  - Can show TSP is also NPC if we can convert any input for HC to an input for TSP in poly time (Why?)



Cycle with cost ? 8 = BDCEB



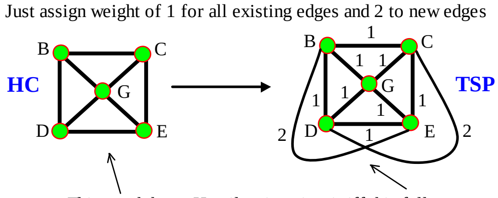
Convert to input for TSP

Input for HC

TSP is NP-complete!

- We can show TSP is also NPC if we can convert any input for HC to an input for TSP in poly time. Here's one way:

Just assign weight of 1 for all existing edges and 2 to new edges



Can prove: This graph has a Hamiltonian circuit iff this <u>fully connected graph</u> has a TSP cycle of total cost ? K = |*V*| (here, K = 5)

Coping with NP-completeness

- Given that it is difficult to find fast algorithms for NPC problems, what do we do?
- Alternatives:
  1. <u>Dynamic programming</u>: Avoid repeatedly solving the same subproblem – use table to store results (see Chap. 10)
  2. <u>Settle for algorithms that are fast on average</u>: Worst case still takes exponential time, but doesn't occur very often
  3. <u>Settle for fast algorithms that give near-optimal solutions</u>: In TSP, may not give the cheapest tour, but maybe good enough
  4. <u>Try to get a "wimpy exponential" time algorithm</u>: It's okay if running time is $O(1.00001^N)$ – bad only for N > 1,000,000