CSE 322
Introduction to Formal Models in Computer Science
Simulating PDA's by CFL's:
An alternate proof of Lemma 3.4.2

Lemma: For each pushdown automaton M there exists a context-free grammar G such that $L(M) = L(G)$.

Proof: We show how to build G from the description of $M = (K, \Sigma, \Gamma, \Delta, s, F)$. We assume without loss of generality that M is of the restricted form we described in class: It's only moves are either blind pushes or pops of single symbols, i.e., every transition in Δ is of the form $((p, a, e), (q, A))$ or $((p, a, A), (q, e))$.

Construction: The basic idea is that for each pair of states $p, q \in K$ we will have a non-terminal B_{pq} in G so that B_{pq} will generate all input strings x that take M from state p to state q so that the resulting stack is unchanged and during the computation the contents already on the stack were never removed. Since those contents were never removed and because we are only doing blind pushes or pops this part of the stack is never even examined during such a computation so we can assume that it is empty.

There is a start symbol S and rules

$$S \to B_{sq}$$

for all states $q \in F$. We have three other kinds of rules:

1. For each $q \in K$, we have the rule
$$B_{qq} \to e$$
2. For each $p, q, r \in K$, we have the rule
$$B_{pr} \to B_{pq}B_{qr}$$
3. For each $p, q, r, t \in K$, symbol $A \in \Gamma$ and $a, b \in \Sigma \cup \{e\}$, if there is a transition $((p, a, e), (q, A))$ and a transition $((r, b, A), (t, e))$ in Δ then we have the rule
$$B_{pt} \to aB_{qr}b$$

The rules of the third type are the only tricky ones. Basically it says that if symbol a is read from the input during the first move (which must be a blind push of some symbol A if it doesn't disturb the current stack) started from the current stack then there must be a corresponding move where that A is first popped to get back down to the original stack (during which some b was read). Furthermore because it was the first pop of the A, that A must never have been affected in the intervening time.