CSE 322: Introduction to Formal Models in Computer Science
Assignment #1
March 30, 1998
due: Monday, April 6

1. (a) Find the positive root ϕ_1 and the negative root ϕ_2 of the quadratic equation $x^2 = x + 1$. (This is worth 0 points, but you need the answer for part (b).)

   (b) The Fibonacci numbers are defined as

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \text{ and} \\ f_{n+2} &= f_n + f_{n+1} \text{ for all nonnegative integers } n. \end{aligned}$$

   Prove that $f_n = (\phi_1^n - \phi_2^n)/\sqrt{5}$, for all nonnegative integers n. (Try this out with a calculator for a few small values of n: it's amazing that it works, since the right hand side of this formula is composed of three irrational numbers.)

2. In this problem you will prove that the rule you learned in elementary school for divisibility by 3 works: a number is divisible by 3 if and only if the sum of its digits is divisible by 3.

   Definition: For any integers x and y, and any positive integer m, $x \equiv y \pmod{m}$ if and only if $x = y + km$ for some integer k.

   (a) Show that, if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$. (This is a very short proof, but you will need the result in part (b).)

   (b) For any nonnegative integer x, let σ_x be the sum of the digits of x when x is written in base 10. Prove by induction on the number of digits of x that $x \equiv \sigma_x \pmod{3}$. (Notice that divisibility by 3 is the special case when $x \equiv \sigma_x \equiv 0 \pmod{3}$.) (Hint: Let $x' \in \{0, 1, \ldots, 9\}^*$ be the representation of the integer x as a decimal string.)

(c) If you are interested, use the ideas in this proof to discover interesting similar divisibility rules, for base 10 and/or base 2.

3. (a) Let L be the language generated by the regular expression $(1 \cup (01) \cup (001))^*$. Let

$$L' = \{e\} \cup \{w \in \{0,1\} \mid w \text{ ends with a 1 and does not contain three consecutive 0's}\}.$$

Prove that $w \in L$ if and only if $w \in L'$, by induction on $|w|$. (That is, $L = L'$.)

(b) Conclude from part (a) that $(1 \cup (01) \cup (001))^*(e \cup 0 \cup (00))$ generates the language of strings over $\{0,1\}$ that do not contain three consecutive 0's.