Powers

$L^2 = L \cdot L$

$L^3 = L \cdot L \cdot L$

$\vdots$

$L^1 = L$

$L^0 = \{\varepsilon\}$

$$\forall n \geq 0 \quad L^n = \begin{cases} L \cdot L^{n-1} & \text{if } n \geq 1 \\ \{\varepsilon\} & \text{if } \cancel{k}\, n = 0 \end{cases}$$

Eg
$\Sigma^2 = \{ w \mid |w| = 2 \}$

$\Sigma^n = \{ w \mid |w| = n \}$

$(\Sigma \cup \{\varepsilon\})^n = \{ w \mid |w| \leq n \}$

Prefix

x is a prefix of w

if $\exists y$ s.t. $w = xy$

E.g.

prefixes of abb are

ε, a, ab, abb

Facts

ε is always a prefix

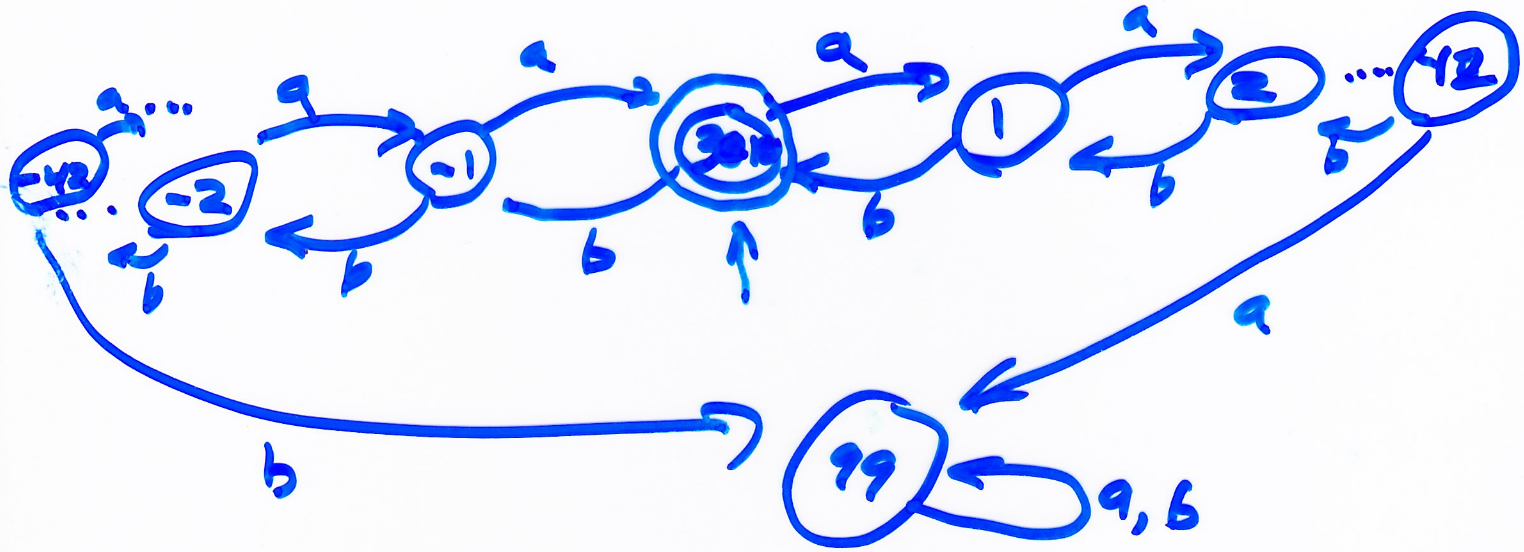
every w is a prefix of itself

if $|w| = n$ then w has $n+1$ prefixes

$\Sigma = \{a, b\}$

$f(w) = \#_a(w) - \#_b(w)$

$L_{eq} = \{w \mid f(w) = 0\}$



$L = \{w \mid f(w) = 0 \ \&\ \forall \text{ prefix } x \text{ of } w \ |f(x)| \leq 42\}$

$g(w) = \begin{cases} f(w) & \text{if } |f(x)| \leq 42 \text{ for all prefixes } x \text{ of } w \\ 99 & \text{o.w.} \end{cases}$

$Q = \{-42, -41, \dots, 41, 42, 99\}$

$q_0 = 0$

$F = \{0\}$

$\delta(q, c) = \begin{cases} q+1 & \text{if } c = a,\ q < 42 \\ q-1 & \text{if } c = b,\ q > -42,\ \neq 99 \\ 99 & \text{o.w.} \end{cases}$

Claim $\forall w \in \Sigma^*$ the state reached by M after reading w is $q = g(w)$

corr. M accepts L

pf M accepts w $\Leftrightarrow$ M ends in F $\Big\}$ by defn + constr.

$\Leftrightarrow$ M ends in O $\Big\}$ by claim

$\Leftrightarrow$ $O = g(w)$ $\Big\}$ by defn.

$\Leftrightarrow$ $w \in L$

Claim $\forall w \in \Sigma^*$, state reached by M after reading w is $g(w)$

$P(n)$: $\forall w \in \Sigma^n$ state .... is $g(w)$

To prove $\forall n \geq 0 \; P(n)$

Basis $n=0$ $\quad w = \epsilon$

M reaches state 0 on ϵ by construction

$g(\epsilon) = 0$ by inspection

say more

Ind $P(n) \Rightarrow P(n+1)$

let w be of length $n+1$

$w = xc$ for some $c \in \Sigma$, $x \in \Sigma^n$

Case 1, $c = a$

(a) $g(x) = 99$

M is in 99 after reading x } by I.H.

$\delta(99, a) = 99$ } by constr

$g(x \cdot a) = 99$ ← argue based on $g(x) \geq 99$

$\therefore P(n+1)$

(b) $g(x) = 42$

.... similar

(c) $-42 \le g(x) < 42$

Min $g(x)$ after x IH

$\delta(g(x), a) = g(x) + 1$ count

$g(xa) = g(x)+1$

$\therefore P(x+1)$

$g(x) < 42$

$\therefore f(x) < 42$

$f(xa) = f(x) + 1 \le 42$