CSE 322
Intro to Formal Models in CS
Homework #2 (Rev b)
Due: Friday, 15 Jan

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Please place problem 4 on separate sheet(s) and turn in separately from 1-3.

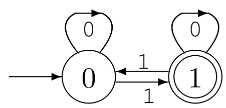
1. Using the following definition of string length,

$$|x| = \begin{cases} 0 & \text{if } x = \epsilon \\ |y| + 1 & \text{if } x = ya \text{ for some } y \in \Sigma^* \text{ and } a \in \Sigma. \end{cases}$$

   prove, by induction on $|v|$, that

$$\forall\, u, v \in \Sigma^*,\ |uv| = |u| + |v|.$$

2. For the DFA below, prove for all $i \in Q$ and $w \in \Sigma^*$ that M is in state i after reading w if and only if $\#_1(w) \equiv i \pmod 2$, where $\#_1(w)$ is the number of 1's in the string w. Prove that $L(M) = \{w \mid \#_1(w) \equiv 1 \pmod 2\}$.



3. Let $\Sigma = \{0, 1\}$. For any string $w \in \Sigma^*$, define a function from $b : \Sigma^* \to \mathbb{N}$ (the natural numbers) as follows:

$$b(w) = \begin{cases} 0 & \text{if } w = \epsilon \\ 2 * b(x) + a & \text{if } w = xa \text{ for some } x \in \Sigma^* \text{ and } a \in \Sigma. \end{cases}$$

   (In the second case of the definition, the "+a" is treating "a" as an integer in the obvious way, rather than as a character from Σ.)

   (a) What are $b(1), b(10), b(100), b(1001), b(10011)$? Say in simple English what the function b is. (I want a non-algorithmic description.)

   (b) Let $M = (Q, \Sigma, \delta, 0, \{0\})$ where $Q = \{0, 1, \ldots, 4\}$ and for all $q \in Q, a \in \Sigma, \delta(q, a) = (2 * q + a) \bmod 5$. Draw M's state diagram.

   (c) Give the sequences of states M is in while reading 10011.

   (d) Give a concise, mathematical statement characterizing what state M is in after reading any given string w. E.g., "M is in state i iff $42 * \#_1(w) \equiv i \pmod 5$" is an (incorrect) example of what such a statement might look like.

   (e) Prove it, by induction on $|w|$.

   (f) Based on (d,e), what is $L(M)$?

4. (a) Give a formal inductive proof of the key claim needed to establish the correctness of the "Cartesian product construction" used in Theorem 1.25 (1st ed.: 1.12): For all $x \in \Sigma^*$, and all $r_1 \in Q_1, r_2 \in Q_2$, we will have M in state (r_1, r_2) after reading x if and only if M_i is in state r_i after reading x, for $i = 1, 2$.

   (b) Then use this fact to prove that $L(M) = L(M_1) \cup L(M_2)$.

   (c) Modify the construction of M slightly, giving a DFA M' accepting $L(M_1) - L(M_2)$ (i.e., the set of strings in $L(M_1)$ but not in $L(M_2)$). Use part (a) to prove your construction correct (i.e., that $L(M') = L(M_1) - L(M_2)$).