CSE 322, Fall 2010

Regular Expressions

Regular expressions over Σ

$\emptyset$ is an r.e.

ε .... r.e.

a ..... for each $a \in \Sigma$

if R_1 & R_2 are r.e.s,
then so are

$(R_1 \cup R_2)$

$(R_1 \circ R_2)$

$(R_1{}^*)$

---

the language denoted by R, $L(R)$ is:

$L(\emptyset) = \emptyset$

$L(\varepsilon) = \{\varepsilon\}$

$\vdots$

$L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$

$$L(\underbrace{(\phi^*)}_{RE}) = L(\phi)^*$$

$$= \phi^*$$

$$= \{\varepsilon\}$$

Short hands

$$\Sigma = \{a, b, c\}$$

$$L(((a \cup b) \cup c)) = \Sigma$$

$$(\underline{\Sigma}^* \cup \varepsilon) \cdot a$$

$$(((\underline{(a \cup b) \cup c)^*} \cup \varepsilon) \cdot a)$$

precedence & associativity

$$(a \cup b \cup c)$$

$$a \cup b \cdot c^*$$

$$(a \cup (b \cdot (c^*)))$$

"words ending with ".TXT" "

$\Sigma^*.TXT$

$(a \cup b \cup \dots \cup z) \cdot (a \cup \dots \cup z \cup 0 \dots 9)^*$

$l \cdot (l \cup d)^*$

Shorthand

$(\Sigma\Sigma)^*$

$\left.\begin{array}{l} \Sigma\Sigma^* \\ aa^* \end{array}\right\} \begin{array}{l} \Sigma^+ \\ a^+ \end{array}$

0^*10^*

$(\varepsilon \cup \Sigma)(\varepsilon \cup \Sigma)$

$\Sigma\Sigma$

$00 \in \quad 0^*(10^*10^*)^*$

$00 \notin \quad (0^*10^*10^*)^*$

$(0^*10^*1)^*0^*$

$(d^*.d^+ \cup d^+.d^*)(\varepsilon \cup E(\varepsilon \cup + \cup -)d^+)$

Theorem:

$\forall$ regular expression R $\exists$ an NFA M_R st $L(R) = L(M_R)$

Proof:

By induction on K, the # of $\cup, \circ, *$ operators in R

Base cases ($K=0$):

Then R is "ϕ", "ε", or "a" for $a \in \Sigma$.

Explicitly give simple NFA's recognizing ϕ, $\{\varepsilon\}$, and $\{a\}$ for each $a \in \Sigma$.

(details omitted)

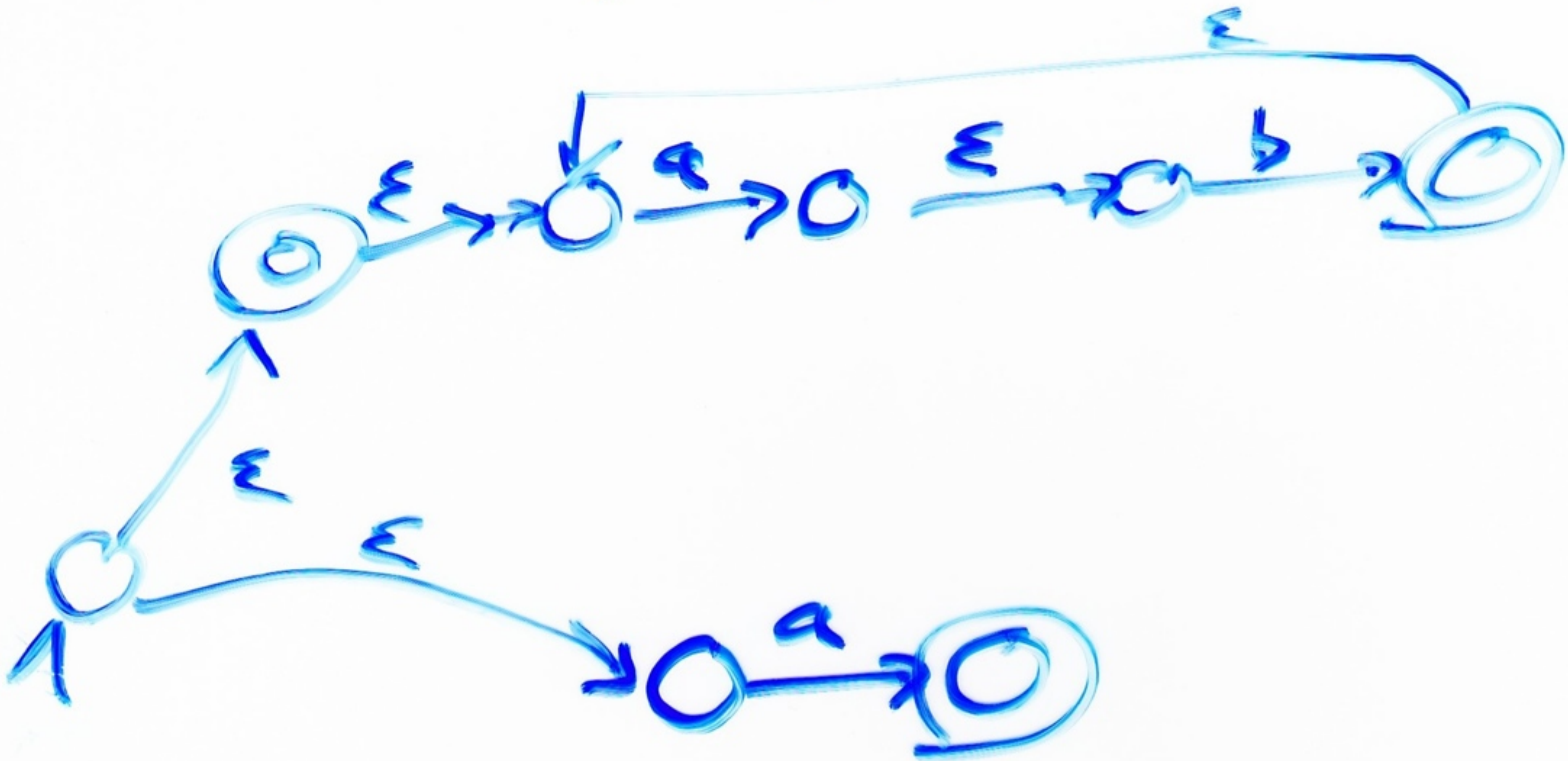
Induction Step (R has $K>0$ operators)

I.H.: assume that for all regular expressions R' with $< K$ operators, $\exists$ NFA $M_{R'}$ recognizing $L(R')$

R has $K>0$ operators. So R is $(R_1 \cup R_2)$ or $(R_1 \circ R_2)$ or $(R_1)^*$ where R_1 (& R_2 if any) have $\leq K-1$ operators. By I.H., $\exists M_{R_1}$ (& M_{R_2}) st. $L(R_i) = L(M_{R_i})$, $i=1,2$. Modify/join it/them as in previous proofs of closure under $\cup, \circ, *$ to get M_R st. $L(R) = L(M_R)$.

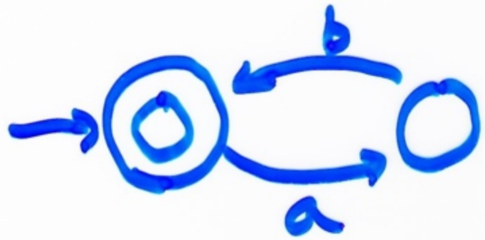
Example

$(ab)^* \cup a$



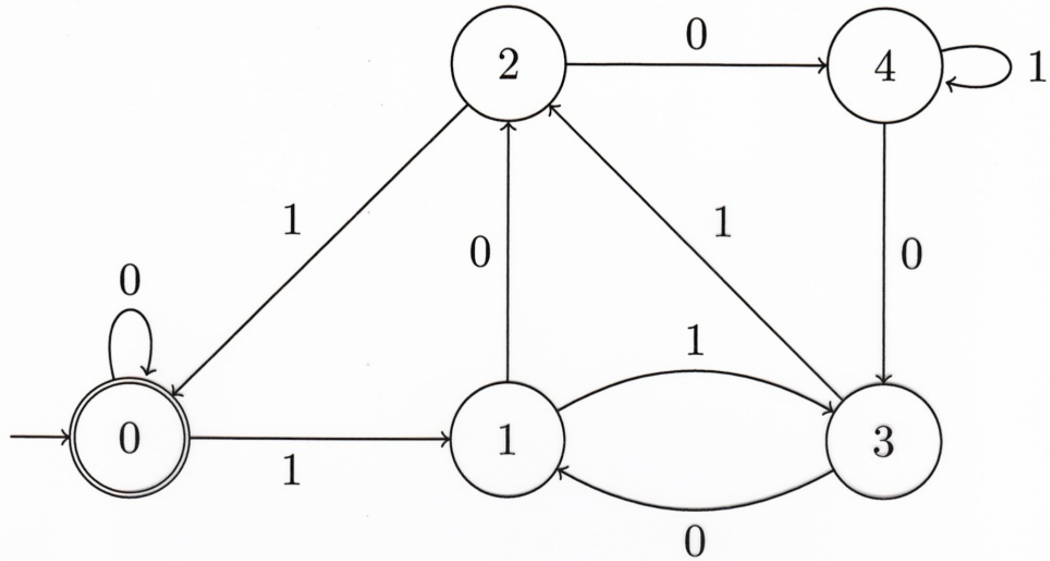
Converse?

For every D/NFA $\exists$ reg expr defining the same language.



$\rightarrow \quad (ab)^*$

$\xrightarrow{?}$

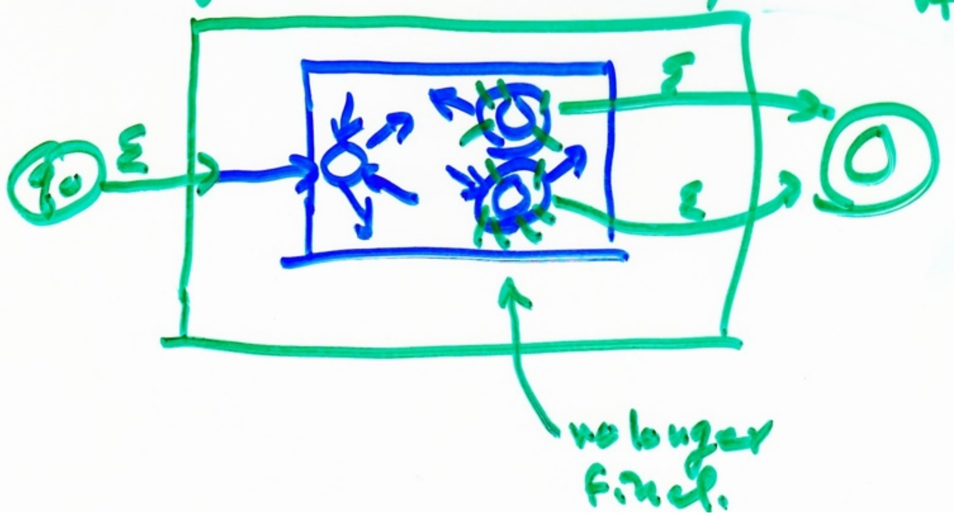


pattern?

0
101
1010
1111
10100
11001
11110
100011
101000
101101
110010
110111
111100
⋮

Every regular language can be described by a regular expression

Note: No loss in assuming no edges into q_0 / out of F / only one $q_f \in F$



GNFA

$G = (Q, \Sigma, \delta, q_0, q_f)$

$Q, \Sigma, q_0, q_f \in Q$ as usual

Regular expressions over Σ

$\delta : (Q - \{q_f\}) \times (Q - \{q_0\}) \to R_\Sigma$

Defn

- G can be in state $q \in Q$ after reading $x \in \Sigma^*$ if $\exists k \geq 0$,

  $\exists\ r_0, r_1, \dots, r_k \in Q$

  $\exists\ x_1, \dots, x_k \in \Sigma^*$

  such that

  (i) $x = x_1 \cdot x_2 \cdot \dots \cdot x_k$

  (ii) $r_0 = q_0$

  (ii) $r_k = q$

  (iii) $\forall 1 \leq i \leq k,\ x_i \in L(\delta(r_{i-1}, r_i))$

- $L(G) = \{x \mid G$ can be in state $q_f \dots \}$

Note: δ syntax a little different; maps state pair to label (reg. exp.) rather than state $\times$ symbol $\to$ new state.

Theorem

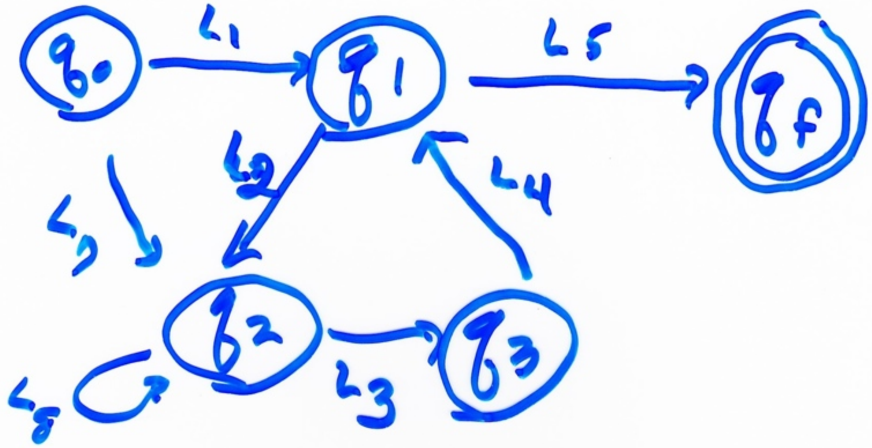
If L is accepted by a GNFA, then L is regular

Pf sketch:

replace edge labeled "r" by NFA equivalent to r based on previous theorem.

If L is regular, then L=L(R) for some regular expression R

Proof will take FA for L, & reduce it to a (G)NFA for same L with progressively fewer states until R becomes obvious.



Q: What strings accepted by

$q_0 \to q_1 \to q_f$?

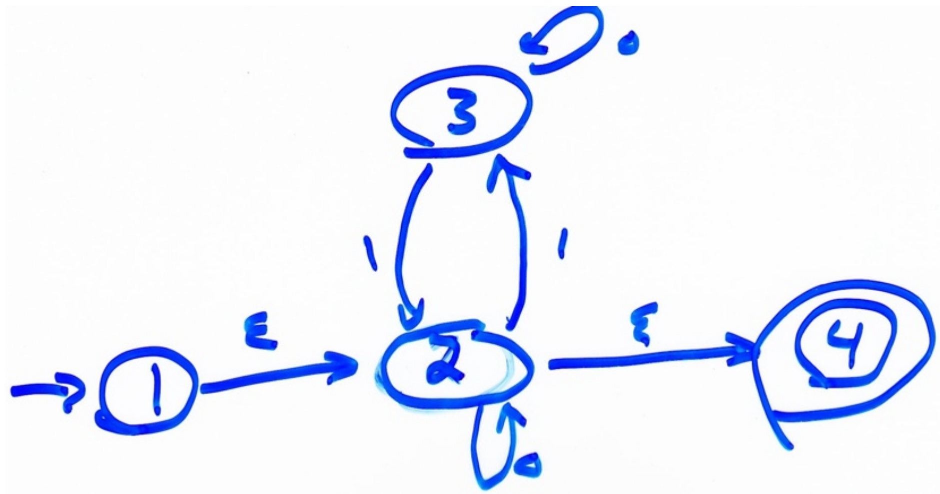
$\{ w \mid w = x_1 x_2 \text{ with } x_1 \in L_1 \ \& \ x_2 \in L_5 \}$

$= L_1 \circ L_5$

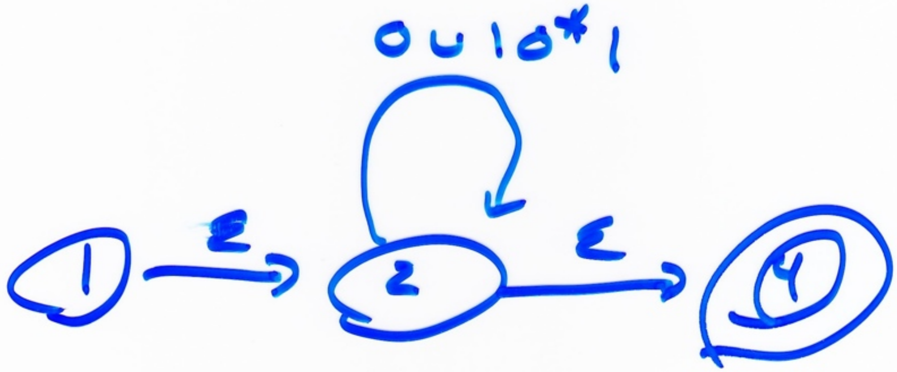
$q_0 \to q_1 \to q_2 \to q_3 \to q_1 \to q_f$

$L_1 \circ L_2 \circ L_3 \circ L_4 \circ L_5$

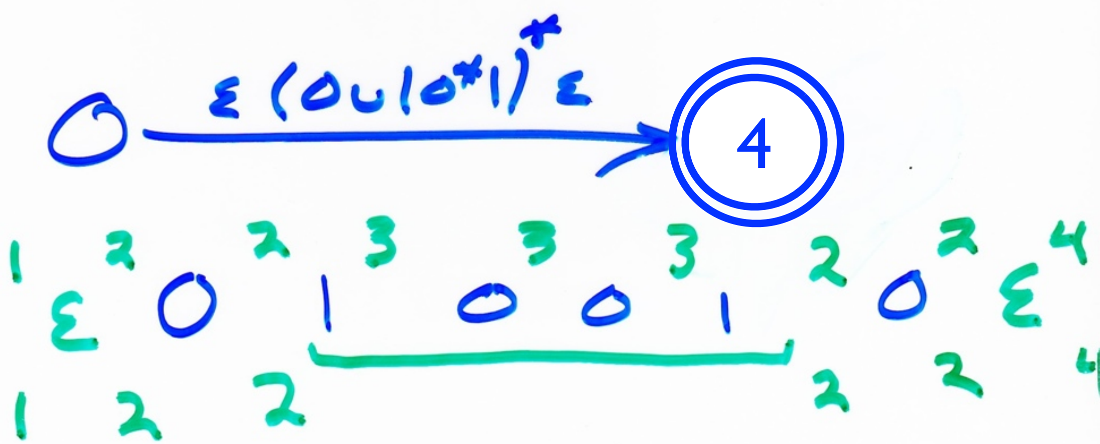
$L = \bigcup_{\text{paths } p}$ concat of L_i's on path p



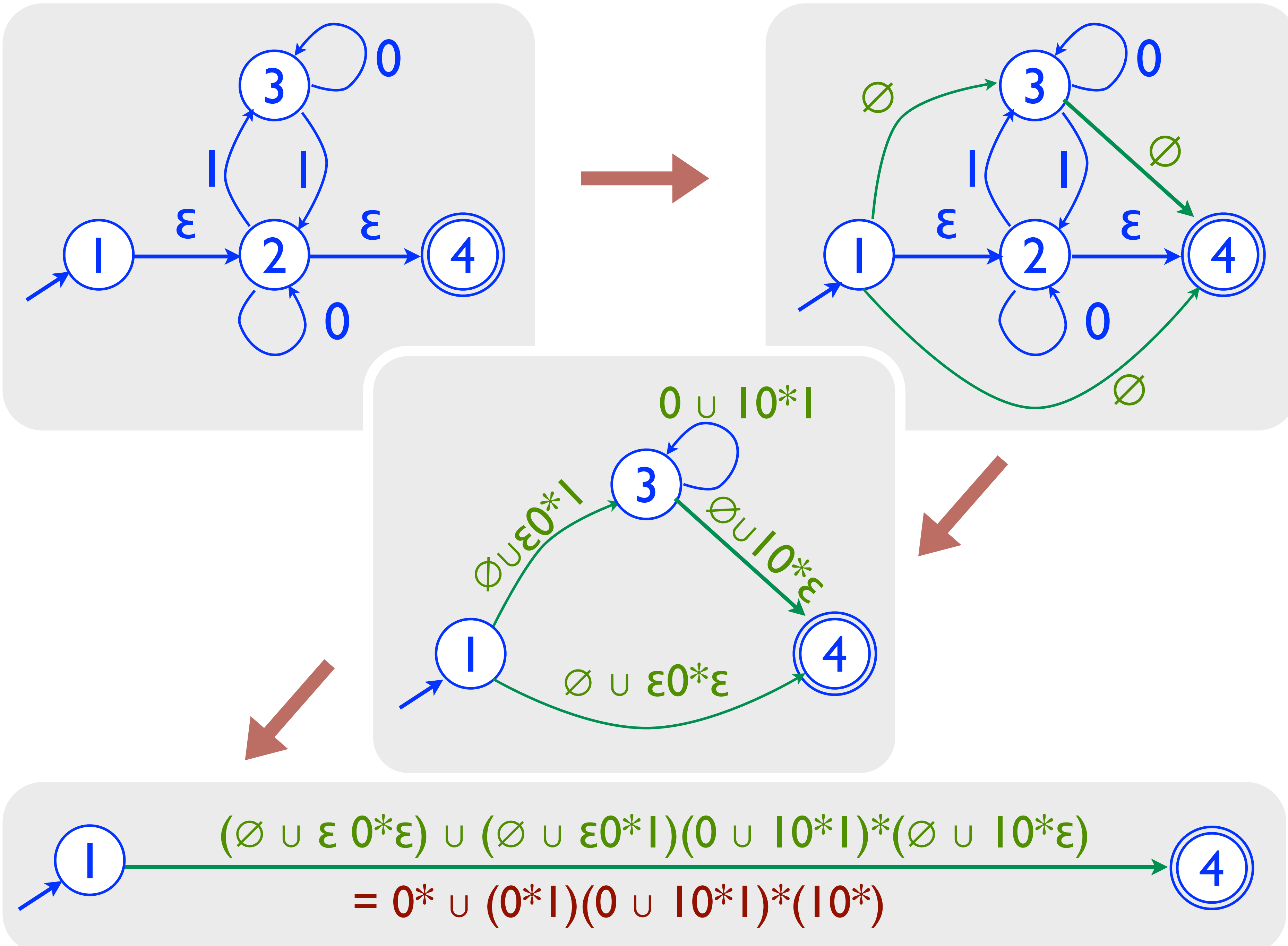
0
1
1
ε
ε
0



0 ∪ 10*1
ε
ε



ε(0∪10*1)*ε
4
ε 0 1 0 0 1 0 ε
1 2 2 2 2 4



3
0
1
1
ε
ε
1
2
4
0
∅
∅
∅
0 ∪ 10*1
∅∪ε0*1
∅∪10*ε
∅ ∪ ε0*ε
(∅ ∪ ε 0*ε) ∪ (∅ ∪ ε0*1)(0 ∪ 10*1)*(∅ ∪ 10*ε)
= 0* ∪ (0*1)(0 ∪ 10*1)*(10*)

Given GNFA $G = (Q, \Sigma, \delta, q_0, q_f)$
With > 2 states

"the old machine"

Notation $\forall\, q_i \neq q_f,\ q_j \neq q_0$

$$\boxed{r_{ij} = \delta(q_i, q_j)}$$

Pick anystate $q_k \neq q_0, q_f$

Build GNFA $G' = (Q', \Sigma, \delta', q_0, q_f)$
with one less state as follows:

"the new machine"

$Q' = Q - \{q_k\}$

$$\boxed{\delta'(q_i, q_j) = r'_{ij} = r_{ij} \cup r_{ik}\, r_{kk}^{*}\, r_{kj}}$$

Claim! G & G' are equivalent

To prove this, it is useful to focus on a sub problem: how do edges in G' relate to paths in G?

In a nutshell, delete state k from G, but enlarge language on each edge to compensate, so that potential contribution of k is added to each edge in G’

```
               r   s   u   v   w   x   y   z
Path in G:   1   2   3   k   4   k   k   2   1 ...

Path in G’:  1   2   3       4           2   1 ...
               r    s    uv       wxy      z
```

strings in
L(edge reg exp)

Relating edges of G' to paths of G

A path in G : any sequence of states

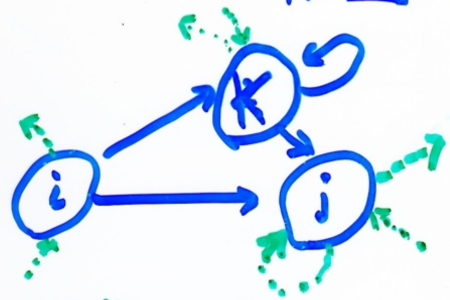
A simple path in G : any sequence of ≥ 2 states s.t. 1^{st} & last are not k, and all intermediate ones (if any) are k.

$i \to j$

$i \to k \to j$

$i \to k \to k \to j$

$\vdots$



other edges ignored/forbidden

The Point:

(a) every path in G can be decomposed into simple paths

(b) every edge in G', say $i \to j$, corresponds to the set of all simple paths in G with those endpoints

Claim 2

$L(r'_{ij}) = \{ w \mid G$ can move from i to j reading w and passing through no intermediate states except possibly k. $\}$

Equivalently:

$L(r'_{ij}) = \{ w \mid G$ can move from i to j reading w along a simple path $\}$

$\stackrel{?}{=} L(r_{ij} \cup r_{ik} \cdot r_{kk}^{*} \cdot r_{kj})$

Claim 4 $\forall$ NFA $\exists$ equiv. reg. expr.

Proof: NFA $\rightarrow$ GNFA $\rightarrow$ 2-state GNFA $\rightarrow$ r.e.

by induction on k, using claim 1

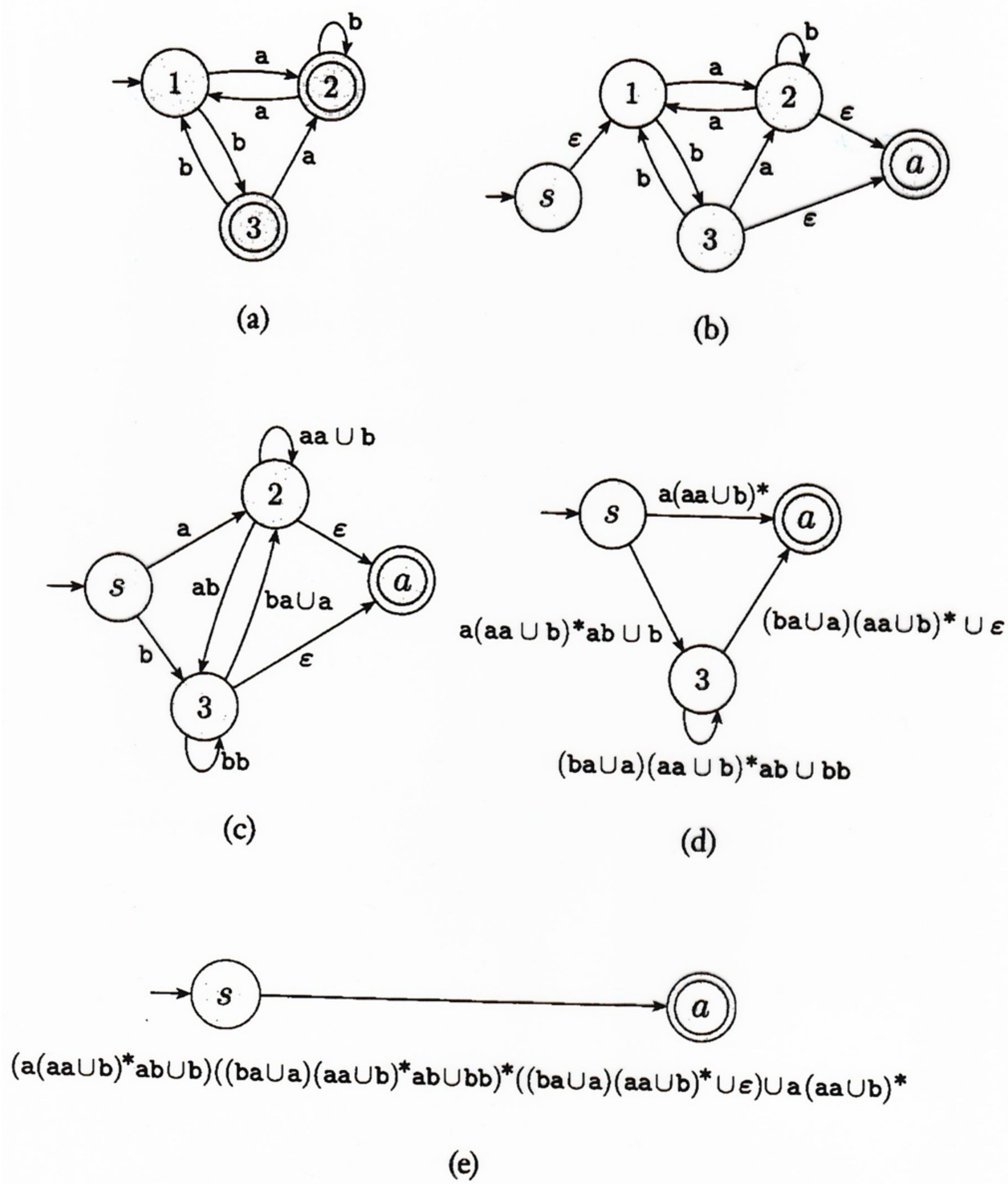

1
2
3
a
a
b
b
b
a
(a)
s
1
2
3
a
ε
a
a
b
b
b
a
ε
ε
(b)
s
2
3
a
aa ∪ b
a
ab
ba∪a
b
ε
ε
bb
(c)
s
a
3
a(aa∪b)*
a(aa ∪ b)*ab ∪ b
(ba∪a)(aa∪b)* ∪ ε
(ba∪a)(aa ∪ b)*ab ∪ bb
(d)
s
a
(a(aa∪b)*ab∪b)((ba∪a)(aa∪b)*ab∪bb)*((ba∪a)(aa∪b)*∪ε)∪a(aa∪b)*
(e)

FIGURE 1.69

Summary

L is regular $\iff$

$L = L(M)$ for some DFA M

$\iff$ $L = L(N)$ . . . . NFA N

$\iff$ $L = L(G)$ . . . . GNFA G

$\iff$ $L = L(R)$ . . . . . . reg. exp. R