## Are There Languages That Are Not Even Recognizable?

- Recall from last class:
  $A_{TM} = \{<M,w> \mid M \text{ is a TM and } \mathbf{M\ accepts\ w}\}$
  $A_H = \{<M,w> \mid M \text{ is a TM and } \mathbf{M\ halts\ on\ w}\}$
- $A_{TM}$ and $A_H$ are undecidable but Turing-recognizable
  - Are there languages that are not even Turing-recognizable?
- What happens if a language A and its complement $\overline{A}$ are both Turing-recognizable?



## Are There Languages That Are Not Even Recognizable?

- **What happens if both A and $\overline{A}$ are Turing-recognizable?**
  - There exist TMs M1 and M2 that recognize A and $\overline{A}$
  - **Can construct a decider for A!** On input w:
  1. Simulate M1 and M2 on w one step at a time, alternating between them.
  2. If M1 accepts, then ACC w and halt; if M2 accepts, REJ w and halt.
- Thm: A and $\overline{A}$ are both Turing-recognizable iff A is decidable
- **Corollary: $\overline{A}_{TM}$ and $\overline{A}_H$ are not Turing-recognizable**
  - If they were, then $A_{TM}$ and $A_H$ would be decidable

# The Chomsky Hierarchy of Languages

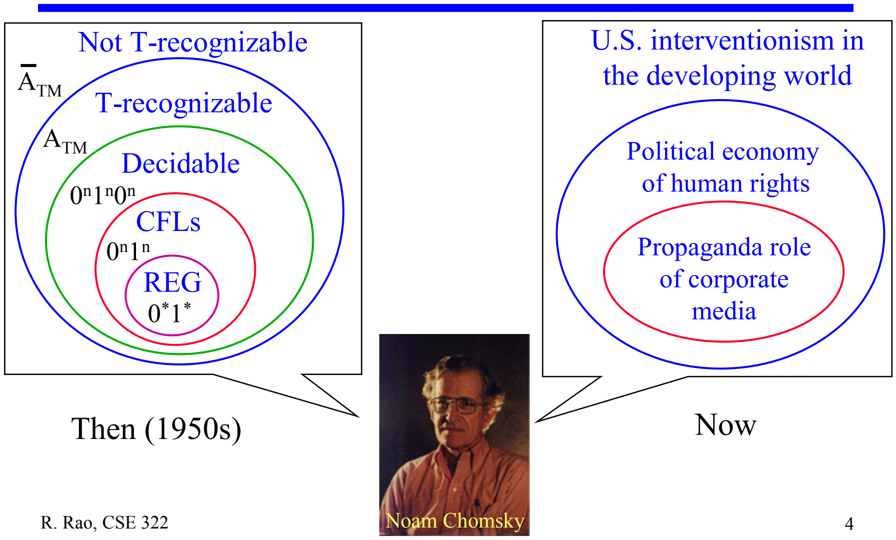
Increasing generality →

| **Language** | Regular | Context-Free | Decidable | Turing-Recognizable |
|---|---|---|---|---|
| **Computational Models** | DFA, NFA, RegExp | PDA, CFG | Deciders – TMs that halt for all inputs | TMs that may loop for strings not in language |
| **Examples** | $(0\cup1)^*11$ | $\{0^n1^n \mid n \geq 0\}$, $\{ww^R \mid w \in \{0,1\}^*\}$ | $\{0^n1^n0^n \mid n \geq 0\}$, $A_{DFA}$, $A_{CFG}$ | $A_{TM}$, $A_H$, $E_{TM}$ |

(Chomsky also studied context-sensitive languages (CSLs, e.g. $a^nb^nc^n$), a subset of decidable languages recognized by linear-bounded automata (LBA))



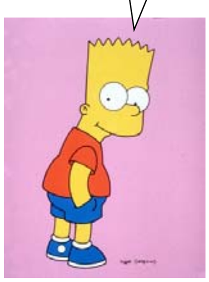
# The Chomsky Hierarchy – Then & Now…



Review Slides for the Final Exam are 3 slides away…



**This space for rent**



## Final Exam

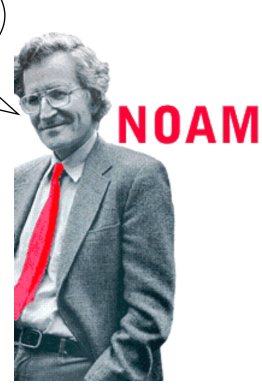
- Details regarding the Final Exam
  - When: Monday, March 13, 2006 from 2:30-4:20 p.m.
  - Where: Same classroom
  - What will it cover?
    - <u>Chapters 0-4 and Chapter 5 up to Thm. 5.4.</u>
    - Emphasis will be on material covered after midterm (Chapter 2 and beyond)
    - You may bring 1 page of notes (8 ½” x 11” sheet!)
      - Plus your midterm page of notes (if you wish)
    - Approximately 6 questions
  - How do I ace it?
    - Practice, practice, practice!
    - See class website for sample final exam and solutions



I believe the Final exam will be decidable!

Stay cool with da pumpin’ lemma!

I believe the world’s problems are politically decidable.




I believe my next movie will be unrecognizable.

## Review of Chapters 0-1

- See Midterm Review Slides
  - Emphasis on:
    - Sets, strings, and languages
    - Operations on strings/languages (concat, *, union, etc)
    - Lexicographic ordering of strings
    - DFAs and NFAs: definitions and how they work
    - Regular languages and properties
    - Regular expressions and GNFAs (see lecture slides)
    - Pumping lemma for regular languages and showing nonregularity



## Context-Free Grammars (CFGs)

- CFG $G = (V, \Sigma, R, S)$
  - Variables, Terminals, Rules, Start variable
  - uAv yields uwv if $A \rightarrow w$ is a rule in G: Written as $uAv \Rightarrow uwv$
  - $u \Rightarrow^* v$ if u yields v in 0, 1, or more steps
  - $L(G) = \{w \mid S \Rightarrow^* w\}$
  - CFGs for regular languages: Convert DFA to a CFG (Create variables for states and rules to simulate transitions)
- Ambiguity: Grammar G is ambiguous if G has two or more parse trees for some string w in L(G)
  - See lecture notes/text/homework for examples
- Closure properties of Context-Free languages
  - Closed under $\cup$, concat, * *but not* $\cap$ *or complementation*.
  - See homework and lecture slides

## Pushdown Automata (PDA)

- PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$
  - $Q$ = set of states
  - $\Sigma$ = input alphabet
  - $\Gamma$ = stack alphabet
  - $q_0$ = start state
  - $F \subseteq Q$ = set of accept states
  - Transition function $\delta: Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to Pow(Q \times \Gamma_\varepsilon)$
  - (current state, next input symbol, popped symbol) $\to$ {set of (next state, pushed symbol)}
  - Input/popped/pushed symbol can be $\varepsilon$
- Example PDAs for:
  - $\{w\#w^R \mid w \in \{0,1\}^*\}$, $\{ww^R \mid w \in \{0,1\}^*\}$



## Context-Free Languages: Main Results

- CFGs and PDAs are equivalent in computational power
  - Generate/recognize the same class of languages (CFLs)
  1. If L = L(G) for some CFG G, then L = L(M) for some PDA M
     - Know how to convert a given CFG to a PDA
  2. If L = L(M) for some PDA M, then L = L(G) for some CFG G
     - Be familiar with the construction – no need to memorize the induction proof
- Pumping Lemma for CFLs
  - Know the exact statement: L CFL $\Rightarrow \exists p$ s.t. $\forall s$ in L s.t. $|s| \geq p$, $\exists\, u, v, x, y$, and $z$ s.t. $s = uvxyz$ and: **1.** $uv^ixy^iz \in L\ \forall\, i \geq 0$, **2.** $|vy| \geq 1$, and **3.** $|vxy| \leq p$.
- Using the PL to show languages are not CFLs
  - E.g. $\{0^n1^n0^n \mid n \geq 0\}$ and $\{0^n \mid n \text{ is a prime number}\}$

## Turing Machines: Definition and Operation

- TM M = $(Q, \Sigma, \Gamma, \delta, q_0, q_{ACC}, q_{REJ})$
  - Q = set of states
  - $\Sigma$ = input alphabet not containing blank symbol "_"
  - $\Gamma$ = tape alphabet containing blank "_", all symbols in $\Sigma$, plus possible temporary variables such as X, Y, etc.
  - $q_0$ = start state
  - $q_{ACC}$ = accept and halt state
  - $q_{REJ}$ = reject and halt state
  - Transition function $\delta$: $Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$
- $\delta$(current state, symbol under the head) = (next state, symbol to write over current symbol, direction of head movement)
  - *Configurations* of a TM, definition of language L(M) of a TM M



## Decidable versus Recognizable Languages

- A language is Turing-recognizable if there is a Turing machine M such that L(M) = L
  - For all strings in L, M halts in state $q_{ACC}$
  - For strings not in L, M may either halt in $q_{REJ}$ or loop forever
- A language is decidable if there is a "decider" Turing machine M that halts on all inputs such that L(M) = L
  - For all strings in L, M halts in state $q_{ACC}$
  - For all strings not in L, M halts in state $q_{REJ}$
- Showing a language is decidable by construction:
  - *Implementation level description of deciders*
  - E.g. $\{0^n1^n0^n \mid n \geq 0\}$, $\{0^n \mid n = m^2$ for some integer m$\}$, see text

## Equivalence of TM Types & Church-Turing Thesis

- Varieties of TMs: Know the definition, operation, and idea behind proof of equivalence with standard TM
  - Multi-Tape TMs: TM with k tapes and k heads
  - Nondeterministic TMs (NTMs)
    - Decider if all branches halt on all inputs
  - Enumerator TM for L: Prints all strings in L (in any order, possibly with repetitions) and only the strings in L
- Can use any of these variants for showing a language is Turing-recognizable or decidable
- Church-Turing Thesis (not a theorem!): Any formal definition of "algorithms" or "programs" is equivalent to Turing machines



## Decidable Problems

- Any problem can be cast as a language membership problem
  - Does DFA D accept input w? Equivalent to: Is <D,w> in $A_{DFA}$ = {<D,w> | D is a DFA that accepts input w}?
- Decidable problems concerning languages and machines:
  - $A_{DFA}$
  - $A_{NFA}$ = {<N,w> | N is a NFA that accepts input w}
  - $A_{REX}$ = {<R,w> | R is a reg. exp. that generates string w}
  - $A_{empty\text{-}DFA}$ = {<D> | D is a DFA and L(D) = ∅}
  - $A_{Equal\text{-}DFA}$ = {<C,D> | C and D are DFAs and L(C) = L(D)}
  - $A_{CFG}$ = {<G,w> | G is a CFG that generates string w}
  - $A_{empty\text{-}CFG}$ = {<G> | G is a CFG and L(G) = ∅}

## Undecidability, Reducibility, Unrecognizability

- $A_{TM}$ = {<M,w> | M is a TM and M accepts w} is Turing-recognizable but not decidable (Proof by diagonalization)
- To show a problem A is undecidable, reduce $A_{TM}$ to A
  - Show that if A was decidable, then you can use the decider for A as a *subroutine* to decide $A_{TM}$
  - E.g. Halting problem = "Does a program halt for an input or go into an infinite loop?"
  - Can show that the Halting problem is undecidable by reducing $A_{TM}$ to $A_H$ = { <M,w> | TM M halts on input w}
- A is decidable iff A and $\overline{A}$ are both Turing-recognizable
  - Corollary: $\overline{A}_{TM}$ and $\overline{A}_H$ are not Turing-recognizable