Recap of Undecidability Proof

- The Question: Are there languages that are not decidable by any Turing machine (TM)?
  - i.e. Are there problems that cannot be solved by any algorithm?
- Consider the language:
  A_{TM} = {<M,w> | M is a TM and M accepts w}
  (Recall that <A,B,…> is just a string encoding the objects A, B, …)
- What can we say about A_{TM}?

A_{TM} is Turing-recognizable

- A_{TM} is Turing-recognizable: Recognizer TM R for A_{TM}:
  On input string <M,w>:
  Simulate M on w.
  ACCEPT <M,w> if M halts & accepts w;
  REJECT <M,w> if M halts & rejects
  (Loop (& thus reject <M,w>) if M ends up looping).
  R accepts <M,w> iff M accepts w, i.e. L(R) = A_{TM}

Is A_{TM} decidable?

- No, A_{TM} = {<M,w> | M is a TM and M accepts w} is undecidable! 1-slide Proof (by Contradiction):
  1. Assume A_{TM} is decidable ⇒ there's a decider H, L(H) = A_{TM}
  2. H on <M,w> = ACC if M accepts w
     REJ if M rejects w (halts in q_{REJ} or loops on w)
  3. Construct new TM D: On input <M>:
     Simulate H on <M,<M>> (here, w = <M>)
     If H accepts, then REJ input <M>
     If H rejects, then ACC input <M>
  4. What happens when D gets <D> as input?
     D rejects <D> if H accepts <D,<D>> if D accepts <D>
     D accepts <D> if H rejects <D,<D>> if D rejects <D>

  Either way: Contradiction! D cannot exist ⇒ H cannot exist
  Therefore, A_{TM} is not a decidable language.

Undecidability Proof uses Diagonalization

Input strings

List of TMs

	<M_1>	<M_2>	<M_3>	…
M_1	ACC	REJ	*loop*	…
M_2	REJ	*loop*	ACC	…
M_3	ACC	ACC	REJ	…
:	:	:	:	

If H exists

D outputs opposite of diagonal

	<M_1>	<M_2>	<M_3>	…	<D>
M_1	ACC	REJ	**REJ**	…	ACC
M_2	REJ	**REJ**	ACC	…	ACC
M_3	ACC	ACC	REJ	…	REJ
:	:	:	:	:	:
D	REJ	ACC	ACC	…	??

D on <M_i> accepts if and only if M_i on <M_i> rejects.
So, D on <D> will accept if and only if D on <D> rejects!
A contradiction ⇒ H cannot exist!
Therefore, A_{TM} is not a decidable language.

One Last Concept: Reducibility

- How do we show a new problem B is undecidable?
- Idea: Show that A_{TM} is reducible to the new problem B
  - What does this mean and how do we show this?
- Show that if B was decidable, then you can use the decider for B as a *subroutine* to decide A_{TM}
  - Contradiction, therefore B must also be undecidable

The Halting Problem is Undecidable (Turing, 1936)

- Halting Problem: Does TM M halt on input w?
  - Equivalent language: A_H = { <M,w> | TM M halts on input w}
  - Need to show A_H is undecidable
  - We know A_{TM} = {<M,w> | TM M accepts w} is undecidable
- Show A_{TM} is reducible to A_H (Theorem 5.1 in text)
  - Suppose A_H is decidable $\Rightarrow$ there's a decider M_H for A_H
  - Then, we can construct a decider D_{TM} for A_{TM}:
    On input <M,w>, run M_H on <M,w>.
    - If M_H rejects, then REJ (this takes care of M looping on w)
    - If M_H accepts, then simulate M on w until M halts
    - If M accepts, then ACC input <M,w>; else REJ

    $L(D_{TM}) = A_{TM} \Rightarrow A_{TM}$ is decidable! Contradiction $\Rightarrow A_H$ is undecidable
- E.g. 2: Show E_{TM} = {<M> | M is a TM and L(M) = $\varnothing$} is undecidable (see Theorem 5.2 in the text)

Are There Languages That Are Not Even Recognizable?

- A_{TM} and A_H are undecidable but Turing-recognizable
  - Are there languages that are not even Turing-recognizable?
- What happens if both A and $\overline{A}$ are Turing-recognizable?
  - There exist TMs M1 and M2 that recognize A and $\overline{A}$
  - Can construct a decider for A! On input w:
  1. Simulate M1 and M2 on w one step at a time, alternating between them.
  2. If M1 accepts, then ACC w and halt; if M2 accepts, REJ w and halt.
- A and $\overline{A}$ are both Turing-recognizable iff A is decidable
- Corollary: $\overline{A}_{TM}$ and $\overline{A}_H$ are not Turing-recognizable
  - If they were, then A_{TM} and A_H would be decidable

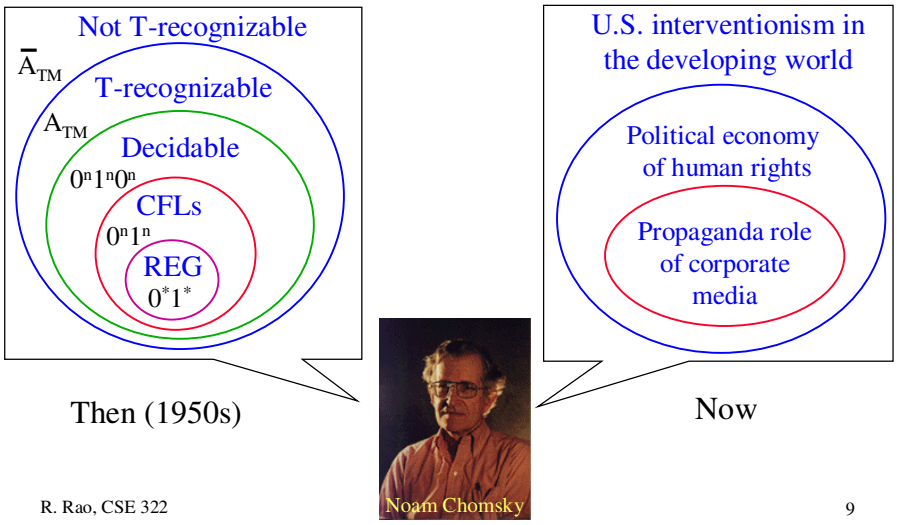
The Chomsky Hierarchy of Languages

Increasing generality →

Language	Regular	Context-Free	Decidable	Turing-Recognizable
Computational Models	DFA, NFA, RegExp	PDA, CFG	Deciders – TMs that halt for all inputs	TMs that may loop for strings not in language
Examples	$(0 \cup 1)^*11$	$\{0^n1^n \mid n \geq 0\}$, Palindromes	$\{0^n1^n0^n \mid n \geq 0\}$, A_{DFA}, A_{CFG}	A_{TM}, A_H

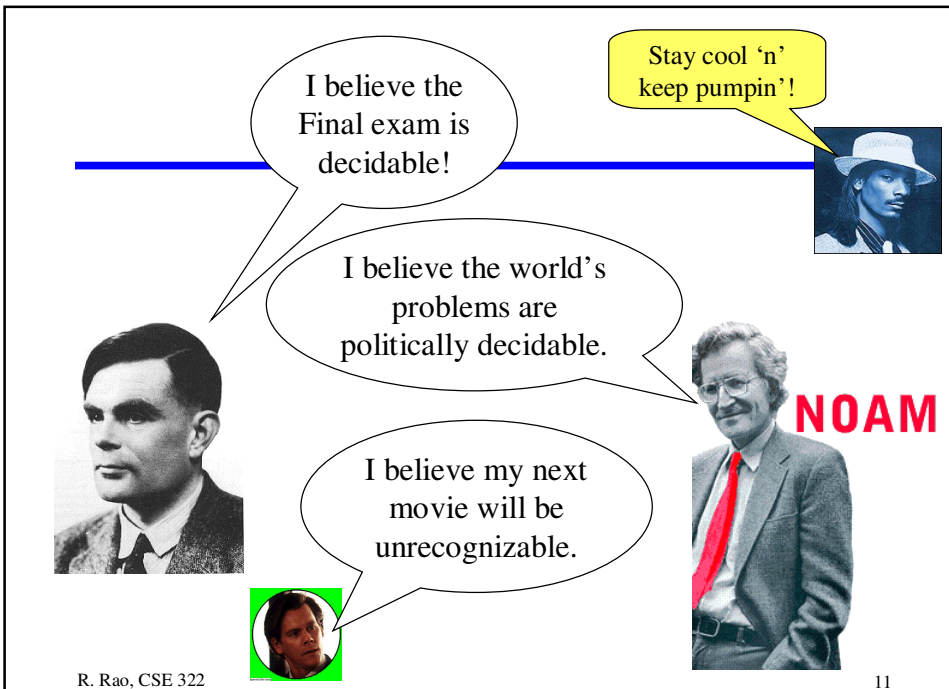
(Chomsky also studied context-sensitive languages (CSLs, e.g. $a^nb^nc^n$) , a subset of decidable languages recognized by linear-bounded automata (LBA))

The Chomsky Hierarchy – Then & Now…



Final Exam

- Details regarding the Final Exam
  - When: Monday, Dec. 16, 2002 from 8:30-10:20 a.m.
  - Where: This classroom EE1 037.
  - What will it cover?
    - Chapters 0-4 and Chapter 5: pages 171-176.
    - Emphasis will be on material covered after midterm (Chapter 2 and beyond)
    - You may bring 1 page of notes (8 ½" x 11" sheet!)
    - Approximately 6 questions
  - How do I ace it?
    - Practice, practice, practice!
    - See class website for sample final exam and solutions


I believe the Final exam is decidable!
Stay cool 'n' keep pumpin'!
I believe the world's problems are politically decidable.
NOAM
I believe my next movie will be unrecognizable.