

Proof of Correctness for NFA to DFA Construction

This proof is taken from *Elements of the Theory of Computation* by Lewis and Papadimitriou, second edition. Any errors are probably due to my faulty transcription.

We are given an NFA $N = (Q, \Sigma, \delta, q_0, F)$. The construction discussed in class produces a DFA $D = (Q', \Sigma, \delta', q'_0, F')$ where $Q' = 2^Q$, $q'_0 = E(q_0)$ and $F = \{q' \in Q' \mid q' \cap F \neq \emptyset\}$ (remember $E(q)$ is the ϵ -closure of q). We need to show that $\mathcal{L}(D) = \mathcal{L}(N)$; we will do this by proving that for any string $w \in \Sigma^*$, $w \in \mathcal{L}(D)$ if and only if $w \in \mathcal{L}(N)$. In other words, if we begin at q'_0 and trace a path through D on the string w , we will be in a final state if and only if there is a path in N starting at q_0 that ends in a final state of N . We will in fact prove something stronger:

Lemma 1 *If we start in any state q of N and trace a path on w , we will end at state p if and only if there is a path in D on w from a state containing $E(q)$ to state P , where P contains p .*

Proof: By induction on $|w|$.

Base Case. Take $|w| = 0$, that is, $w = \epsilon$. We need to show that tracing through N on ϵ , starting from q , leads to p if and only if tracing through D on ϵ , starting from a state containing $E(q)$, leads to P with $p \in P$. As D is deterministic, it can't go anywhere on ϵ , that is, $P = E(q)$. What could p be? Any state reachable in N from q on ϵ , that is, $p \in E(q)$, which equals P . This completes the proof of the base case.

Inductive Hypothesis. Assume the Lemma is true for all strings of length k or less for some $k \geq 0$.

Inductive Step. Consider a string w with $|w| = k + 1$. Let $w = va$, where $v \in \Sigma^*$ and $a \in \Sigma$. Note that $|v| = k$, so the Inductive Hypothesis applies to it.

For the *only if* direction of the proof, we have a path from q to p on w in N . What could this path look like? First, we follow a path on v , and possibly do some ϵ -transitions, to end at state r_1 . Then we read a and get to state r_2 , then we take more ϵ -transitions and get to p . See Figure 1. Now consider a path in D . By the induction hypothesis, after reading v we are in a state R that contains r_1 . By our construction, after reading a we are in a state containing $E(r_2)$. But by the definition of ϵ -closure, $p \in E(a)$.

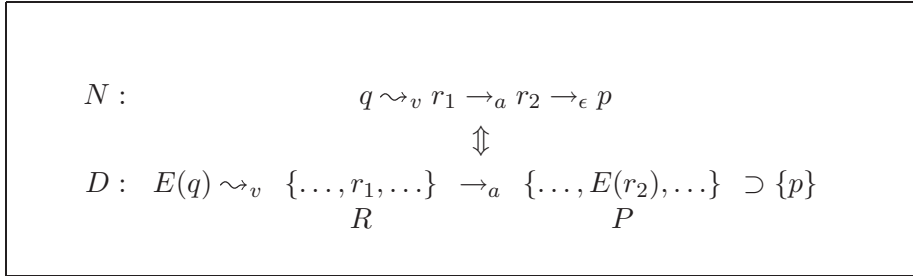


Figure 1: Paths through N and D on $w = va$.

For the *if* direction of the proof, we have a path in D from a state containing $E(q)$ to P . As above, this can be broken down into a path from a state containing $E(q)$ to R on v , then an arc from R to P on a . By the induction hypothesis, there is a path in N from q to some $r_1 \in R$ on v . When we take the transition from r_1 to some other state r_2 on a , and then any ϵ -transitions, by our construction we must be in a state in P .

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