What's on our plate today?

- Cliff's notes for equivalence of CFLs and L(PDAs)
  - L is a CFL $\Rightarrow$ L = L(M) for some PDA M
  - L = L(M) for some PDA M $\Rightarrow$ L = L(G) for some CFG G
- Pumping Lemma (one last time)
  - Statement of Pumping Lemma for CFLs
  - Proof: See class notes from last time and textbook
  - Application: Showing a given L is not a CFL
- Introduction to Turing Machines

From CFLs to PDAs

- L is a CFL $\Rightarrow$ L = L(M) for some PDA M
- Proof Summary:
  - L is a CFL means L = L(G) for some CFG G = (V, Σ, R, S)
  - Construct PDA M = $(Q, \Sigma, \Gamma, \delta, q_0, \{q_{acc}\})$
    M has only 4 main states (plus a few more for pushing strings)
    $Q = \{q_0, q_1, q_2, q_{acc}\} \cup E$ where E are states used in 2 below
  - δ has 4 components:

1. **Init. Stack**: $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$ and $\delta(q_1, \varepsilon, \varepsilon) = \{(q_2, S)\}$
2. **Push & generate strings**: $\delta(q_2, \varepsilon, A) = \{(q_2, w)\}$ for $A \rightarrow w$ in R
3. **Pop & match to input**: $\delta(q_2, a, a) = \{(q_2, \varepsilon)\}$
4. **Accept if stack empty**: $\delta(q_2, \varepsilon, \$) = \{(q_{acc}, \varepsilon)\}$

- Can prove by induction: $w \in L$ iff $w \in L(M)$

From PDAs to CFLs

- $L = L(M)$ for some PDA $M \Rightarrow L = L(G)$ for some CFG G
- Proof Summary: Simulate M's computation using a CFG
  - First, simplify M: 1. Only 1 accept state, 2. M empties stack before accepting, 3. Each transition either Push or Pop, not both or neither. Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{acc}\})$
  - Construct grammar $G = (V, \Sigma, R, S)$
  - Basic Idea: Define variables A_{pq} for simulating M
  - A_{pq} generates all strings w such that w takes M from state p with empty stack to state q with empty stack
  - Then, A_{q0qacc} generates all strings w accepted by M

From PDAs to CFLs (cont.)

- $L = L(M)$ for some PDA $M \Rightarrow L = L(G)$ for some CFG G
- Proof (cont.)
  - Construct grammar $G = (V, \Sigma, R, S)$ where

    $V = \{A_{pq} \mid p, q \in Q\}$

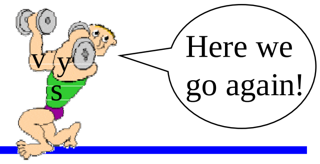
    $S = A_{q0qacc}$

    $R = \{A_{pq} \to aA_{rs}b \mid \; p \xrightarrow{a,\, \varepsilon \to c} r \overset{A_{rs}}{\dashrightarrow} s \xrightarrow{b,\, c \to \varepsilon} q\}$

    $\cup \{A_{pq} \to A_{pr} A_{rq} \mid p, q, r \in Q\}$

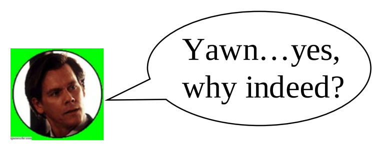
    $\cup \{A_{qq} \to \varepsilon \mid q \in Q\}$
- See text for proof by induction: $w \in L(M)$ iff $w \in L(G)$
- Try to get G from M where $L(M) = \{0^n1^n \mid n \geq 1\}$

Pumping Lemma for CFLs



- Intuition: If L is CF, then some CFG G produces strings in L
  - If some string in L is very long, it will have a very tall parse tree
  - If a parse tree is taller than the number of distinct variables in G, then some variable A repeats $\Rightarrow$ A will have at least two sub-trees
  - We can pump up the original string by replacing A's smaller sub-tree with larger, and pump down by replacing larger with smaller
- Pumping Lemma for CFLs in all its glory:
  - If L is a CFL, then there is a number p (the "pumping length") such that for all strings s in L such that $|s| \geq$ p, there exist u, v, x, y, and z such that $s = uvxyz$ and:
  1. $uv^ixy^iz \in$ L for all $i \geq 0$, and
  2. $|vy| \geq 1$, and
  3. $|vxy| \leq$ p.

Why is the PL useful?



- Can use the pumping lemma to show a language L is *not context-free*
  - <u>5 steps for a proof by contradiction</u>:
  1. Assume L is a CFL.
  2. Let p be the pumping length for L given by the pumping lemma for CFLs.
  3. Choose cleverly an s in L of length at least p, such that
  4. For *all possible ways* of decomposing s into $uvxyz$, where $|vy| \geq 1$ and $|vxy| \leq$ p,
  5. We can choose an $i \geq 0$ such that uv^ixy^iz is not in L.
- In-Class Examples: Show the following are not CFLs
  - $L = \{0^n1^n0^n \mid n \geq 0\}$ and $L = \{0^n \mid n$ is a prime number$\}$

Using the Pumping Lemma

- Show $L = \{0^n \mid n \text{ is a prime number}\}$ is not a CFL
  1. Assume L is a CFL.
  2. Let p be the pumping length for L given by the pumping lemma for CFLs.
  3. Let $s = 0^n$ where n is a prime $\geq p$
  4. Consider *all possible ways* of decomposing *s* into *uvxyz*, where $|vy| \geq 1$ and $|vxy| \leq p$.
     Then, $vy = 0^r$ and $uxz = 0^q$ where $r + q = n$ and $r \geq 1$
  5. We need an $i \geq 0$ such that $uv^ixy^iz = 0^{ir+q}$ is not in L.
     ($i = 0$ won't work because q could be prime: e.g. 2 + 17 = 19)
     Choose $i = (q + 2 + 2r)$. Then, $ir + q = qr + 2r + 2r^2 + q = q(r+1) + 2r(r+1) = (q+2r)(r+1)$ = not prime (since $r \geq 1$).

  So, 0^{ir+q} is not in L $\Rightarrow$ contradicts pumping lemma. L is not a CFL.

Two cool results about CFLs

Oh boy…
Jolly good

- CFLs are not closed under intersection
  - **Proof**: $L_1 = \{0^n1^n0^m \mid n, m \geq 0\}$ and $L_2 = \{0^m1^n0^n \mid n, m \geq 0\}$ are both CFLs but $L_1 \cap L_2 = \{0^n1^n0^n \mid n \geq 0\}$ is not a CFL.
- CFLs are not closed under complementation
  - **Proof by contradiction**:
    Suppose CFLs are closed under complement.

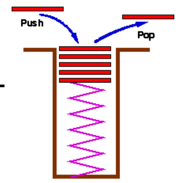
    Then, for L_1, L_2 above, $\overline{\overline{L_1} \cup \overline{L_2}}$ must be a CFL (since CFLs are closed under $\cup$ -- see homework #5, problem 1).

    But, $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$ (by de Morgan's law).

    $L_1 \cap L_2 = \{0^n1^n0^n \mid n \geq 0\}$ is not a CFL $\Rightarrow$ contradiction.

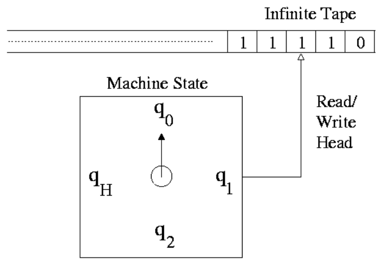
    Therefore CFLs are not closed under complementation.

Can we make PDAs more powerful?

- PDA = NFA + 

- What if we allow arbitrary reads/writes to the stack instead of only push and pop?

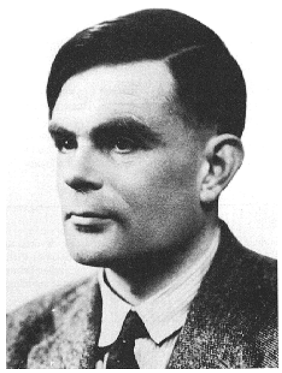
Enter….Turing Machines!



Just like a DFA except:

- You have an infinite "tape" memory (or scratchpad) on which you receive your input and on which you can do your calculations
- You can <u>read</u> 1 symbol at a time from a cell on the tape, <u>write</u> 1 symbol, then <u>move</u> the read/write pointer (head) left (L) or right (R)

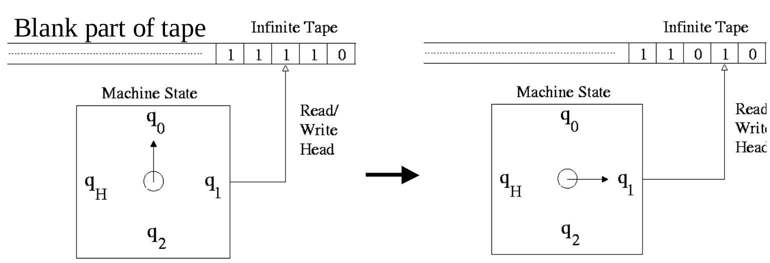
Who's Turing?



- Alan Turing (1912-1954): one of the most brilliant mathematicians of the 20^{th} century (one of the founding "fathers" of computing)
- Click on "Theory Hall of Fame" link on class web under "Lectures"
- Introduced the Turing machine as a formal model of what it means to compute and solve a problem (i.e. an "algorithm")
  - Paper: `On computable numbers, with an application to the Entscheidungsproblem, Proc. London Math. Soc. 42 (1936).`

How do Turing Machines compute?

- δ(current state, symbol under the head) = (next state, symbol to write over current symbol, direction of head movement)



- Diagram shows: $\delta(q_0,1) = (q_1, 0, R)$ (R = right, L = left)
- In terms of "Configurations": $11q_0110 \Rightarrow 110q_110$

Solving Problems with Turing Machines

- We know $L = \{0^n1^n0^n \mid n \geq 0\}$ cannot be accepted by any PDA
- Design a Turing Machine (TM) that accepts L
  - Write down its operation in words…

Next Time…

- Formal definition of TMs
- Solving problems with TMs
- Varieties of TMs (multi-tape, nondeterministic, etc.)
- Homework #5 due on Friday!