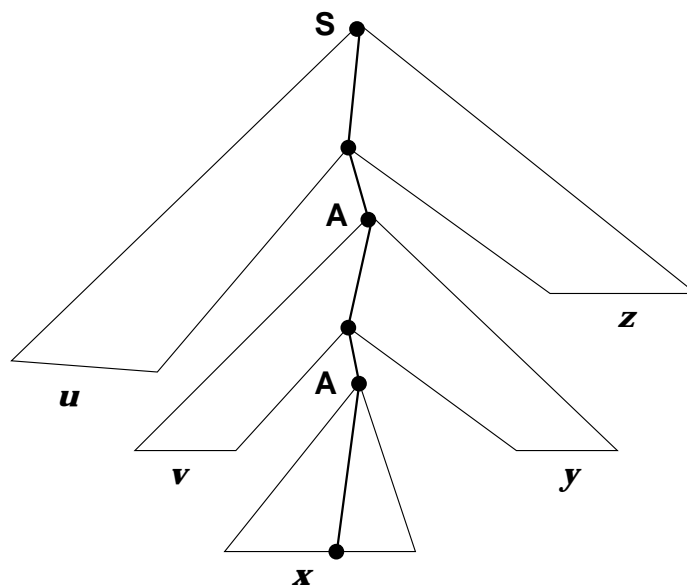


CSE 322  
Introduction to Formal Models in Computer Science  
Pumping Lemma for CFL's

**CFL Pumping Lemma:** If  $L$  is a context-free language then there is a constant  $n$  (depending on  $L$ ) such that for all  $w \in L$  with  $|w| \geq n$ ,  $w$  can be written as  $w = uvxyz$  such that

1.  $|vxy| \leq n$ ,
2.  $|v| + |y| \geq 1$ , and
3. for all  $i \geq 0$ ,  $uv^i xy^i z \in L$ .

**Proof:** Let  $G = (V, \Sigma, R, S)$  be a context-free grammar generating  $L$ . Let  $\ell$  be the length of the longest right side of any rule in  $R$ . Let  $k = |V - \Sigma|$ ,  $n = \ell^{k+1}$  and suppose that  $w \in L$  with  $|w| \geq n$ . Consider the smallest parse tree  $T_w$  for  $w$ . By the choice of  $\ell$  we know that the fan-out of any node in  $T_w$  is at most  $\ell$ . Now any tree of fan-out  $\ell$  and height  $h$  has at most  $\ell^h$  leaves. Therefore we know that  $T_w$  has height at least  $k + 1$  and therefore has some root-leaf path with at least  $k + 1$  interior vertices. Consider the longest such path and consider the labels of the lowest  $k + 1$  interior vertices on this path. All these labels are non-terminals and since  $|V - \Sigma| = k$ , one of these labels must appear twice. Call this non-terminal  $A$ . Thus we can break up  $T_w$  and choose  $u, v, x, y$ , and  $z$  as in the following figure:



We now need to argue that (1), (2), (3) hold.

(1) is true because the higher  $A$  generates  $vw x$  and this  $A$  has height at most  $k + 1$  and therefore the subtree below it has at most  $\ell^{k+1} = n$  leaves.

(2) says that  $v$  and  $y$  are not both  $e$ . This must be true because  $T_w$  was chosen to be minimal to begin with. If  $v = y = e$  then we can get a smaller tree by replacing the subtree below the higher  $A$  by the subtree below the second one.

To show (3) we note that

$$S \xrightarrow[G]{*} uAz \xrightarrow[G]{*} uvAyz \xrightarrow[G]{*} uvxyz = w$$

and this implies that (a)  $A \xrightarrow[G]{*} vAy$  and (b)  $A \xrightarrow[G]{*} x$ . We first show that for all  $i \geq 0$ ,  $S \xrightarrow[G]{*} uv^iAy^iz$  by induction on  $i$  and then conclude that  $S \xrightarrow[G]{*} uv^iwy^iz$  using derivation (b).

Base Case:  $S \xrightarrow[G]{*} uAz = uv^0Ay^0z$ .

Induction Step: Suppose that  $S \xrightarrow[G]{*} uv^iAy^iz$ . Then  $S \xrightarrow[G]{*} uv^iAy^iz \xrightarrow[G]{*} uv^ivAyy^iz = uv^{i+1}Ay^{i+1}z$  using derivation (a).

As a result we have  $uv^iwy^iz \in L$  for all  $i \geq 0$  and (3) is proved.  $\square$

**Slightly Stronger CFL Pumping Lemma:** If  $L$  is a context-free language then there is a constant  $n$  (depending on  $L$ ) such that for all  $w \in L$  with  $|w| \geq n$ ,  $w$  can be written as  $w = uvxyz$  such that

1.  $|vxy| \leq n$ ,
2.  $v \neq e$ , and
3. for all  $i \geq 0$ ,  $uv^ixy^iz \in L$ .

**Proof:** The only difference here is that the condition  $|v| + |y| \geq 1$  is replaced by  $v \neq e$ . Apply the CFL Pumping Lemma above to get  $w = uvxyz$  such that  $|vxy| \leq n$ ,  $|v| + |y| \geq 1$  and  $v$  and  $y$  can be jointly pumped. If  $v \neq e$  we are done. Otherwise if  $v = e$ , then  $w = uxyz$  and pumping  $v$  and  $y$  jointly implies that  $uxy^iz \in L$  for all  $i \geq 0$ . In this case we set  $u_{new} = ux$ ,  $v_{new} = y \neq e$ ,  $x_{new} = e$ ,  $y_{new} = e$ , and  $z_{new} = z$ . It is easy to check now that all the conditions are satisfied for these new values.  $\square$