University of Washington
Department of Computer Science and Engineering
CSE 321, Fall 1994
Handout No. 18

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Final Exam Solutions, Fall, 1994

Problem 1 (50 points):

Give *short* definitions for the following terms. Do not write more than two sentences for each answer.

a) Euler's formula

The formula $v - e + f = 2$, which gives the relationship between the number of vertices, edges, and faces in a connected planar graph.

b) Eulerian Circuit

A cycle in a graph which traverses each edge exactly once.

c) $K_{n,m}$

The complete bipartite graph between a set of size n and a set of size m.

d) Toplogical Sort

An algorithm for constructing a total order which is compatible with a given partial order.

e) Equivalence Relation

A relation which is transitive, symmetric, and reflexive.

f) Lexicographic Order

A total ordering defined on strings or n-tuples. The ordering is made by comparing the strings based upon the first position at which they differ.

g) Composition of Relations

A new relation $S \circ R$ is the composition of R and S where

$$S \circ R = \{(v, w) \mid \exists x \text{ such that } (v, x) \in R \text{ and } (x, w) \in S\}.$$

h) Random Variable

A function which assigns real values to all elements of a sample space.

i) Binomial Coefficient

1. $\binom{n}{k}$ is the number of subsets of size k from a set of size n.
2. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
3. $\binom{n}{k}$ is the k-th entry in the n-th row of Pascal's triangle.

j) Conditional Probability

The probability of an event given the occurence of another event. $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$.

Problem 2 (20 points):

For what values of n does the n-dimensional hypercube (Q_n) contain an Eulerian Circuit? Justify your answer.

Every vertex in the n dimensional hypercube has degree n. A graph has an Eulerian Circuit iff the graph is connected, and every vertex has even degree. Hence, Q_n has an Eulerian Circuit iff n is even.

Problem 3 (20 points):

Let $G = (V, E)$ be an undirected graph. The complement of G, $\overline{G}$ is defined as the graph $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{(v, w) \mid v \in V,\ w \in V,\ v \neq w,\ (v, w) \notin E\}$.

a) (5 points) Recall that C_5 is the cycle of length 5. Draw C_5 and $\overline{C_5}$.

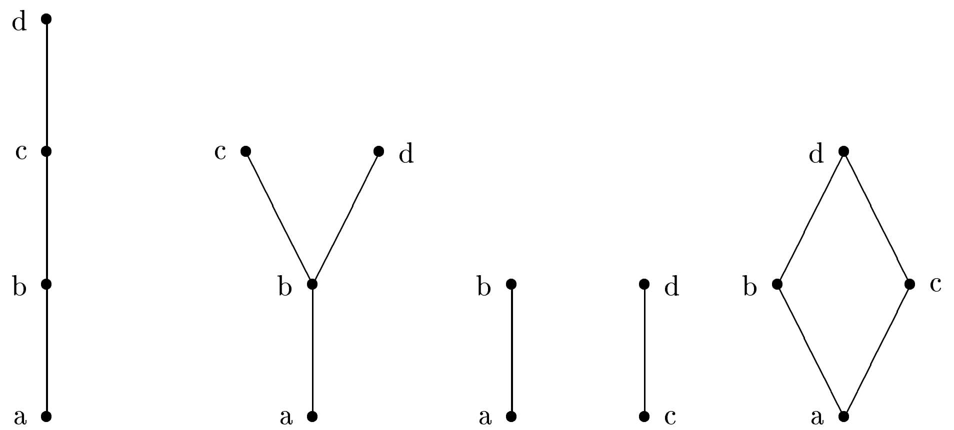
C_5 is a cycle, $\overline{C_5}$ is a star.

b) (15 points) Show that C_5 is isomorphic to $\overline{C_5}$.

If the vertices of C_5 are labeled a, b, c, d and e in cyclic order, then the mapping $f(a) = a$, $f(b) = c$, $f(c) = e$, $f(d) = b$, and $f(e) = d$ is an isomorphism from C_5 to $\overline{C_5}$.

Problem 4 (20 points):

Give the Hasse diagrams for four different ordering relations on the set $\{a, b, c, d\}$.
There are lots of possibilities on this one. Here are four:



Problem 5 (20 points):

Define a relation $x \gg y$ on the positive integers by: $x \gg y$ if $x = y$ or if there exists a $k > 0$ such that $2^k \mid x$ but $2^k \not| y$.
For example, $12 \gg 18$ since $4 \mid 12$ but $4 \not| 18$.

a (10 points) Prove that $\gg$ is a partial order, but not a total order.

i) Reflexive: $x \gg x$ by definition.

ii) Antisymmetric. Suppose $x \gg y$, $x \neq y$. For some k, $2^k \mid x$ but $2^k \not| y$. If $2^j \mid y$, then $j < k$, so $2^j \mid x$, hence $y \not\gg x$.

iii) Transitive: $x \gg y$ and $y \gg z$. For some k_1, $2^{k_1} \mid x$, but $2^{k_1} \not| y$, and for some k_2, $2^{k_2} \mid y$, but $2^{k_2} \not| z$. Since $k_1 > k_2$, $2^{k_1} \not| z$, so $x \gg z$.

$\gg$ is a not a total order, since there are many incomparable pairs, for example, $1 \not\gg 3$ and $3 \not\gg 1$. If m and n are odd, then $2^k m$ and $2^k n$ are incomparable.

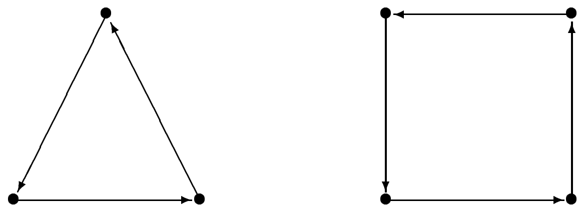
b (10 points) Describe a total order on the positive integers which is compatible with $\gg$.

To extend $\gg$ to a total order, we need to specify the ordering of the incomparible pairs. We define a new order relation $\ggg$: for $x = 2^{k_1} n$, n odd, and $y = 2^{k_2} m$, m odd, $x \ggg y$ if $k_1 > k_2$ or if $k_1 = k_2$ and $n \geq m$.

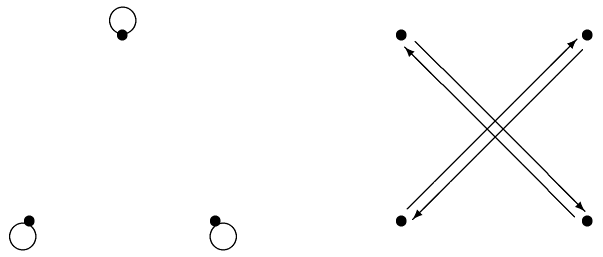
The condition $k_1 > k_2$ ensures that it is compatible with $\gg$. Since any integer can be expressed in the form $2^k m$, with m odd, the relation is total.

Problem 6 (20 points):

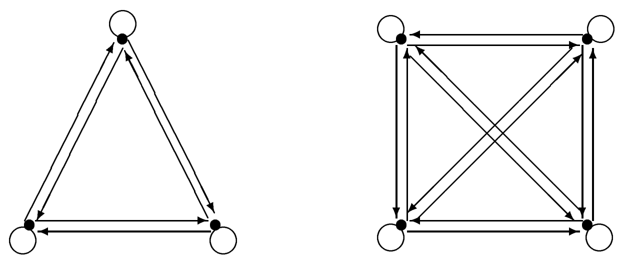
Consider the relation R defined by the following directed graph:



a) (10 points) Draw the relation R^6.



b) (10 points) Draw the transitive closure of R.



Problem 7 (20 points):

Suppose that you are taking a true-false test that contains the statements P and Q. You do not have a clue as to the truth values of P and Q, but you do know that P implies Q. Give a strategy that ensures that you get at least one of the answers correct and explain why your strategy works.

The simple strategy is to guess that P is false, and Q is true. This guarantees at least one correct answer, since if P is false, P was evaluated correctly, and if P is true, Q is also true, so Q was evaluated corrrectly.

Problem 8 (20 points):

Give an induction proof to show that for all $n \geq 1$:

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

Basis: $\sum_{j=1}^{1}(2j-1) = 1 = 1^2$.

Induction: Suppose that

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

So,

$$\begin{aligned}\sum_{j=1}^{n+1}(2j-1) &= 2(n+1)-1+\sum_{j=1}^{n}(2j-1) \\ &= 2(n+1)-1+n^2 \\ &= n^2+2n+1 \\ &= (n+1)^2\end{aligned}$$

Problem 9 (20 points):

Let $t(R)$ denote the transitive closure of R. Give an example of relations R_1 and R_2 over the set $A = \{a, b, c, d\}$ such that

$$t(R_1 \cup R_2) \neq t(R_1) \cup t(R_2).$$

Justify your answer.

The easiest example is $R_1 = \{\langle a,\ b\rangle\}$, $R_2 = \{\langle b,\ c\rangle\}$. So $t(R_1) = \{\langle a,\ b\rangle\}$, $t(R_2) = \{\langle b,\ c\rangle\}$. So $t(R_1) \cup t(R_2) = \{\langle a,\ b\rangle, \langle b,\ c\rangle\}$. However, $R_1 \cup R_2 = \{\langle a,\ b\rangle, \langle b,\ c\rangle\}$, so $t(R_1 \cup R_2) = \{\langle a,\ b\rangle, \langle a,\ c\rangle, \langle b,\ c\rangle\}$.

Problem 10 (20 points):

A computer system requires passwords of length 6, each of which must contain exactly 3 digits (from 0 to 9) and exactly 3 lower case letters. Characters may be repeated, and the letters and digits may occur in any order, so "a37xa9" is a valid password.

What is the total number of valid passwords? You may express your answer as a formula involving binomial coefficients.

If we fix a pattern of letters, and digits, say $LLLDDD$, there are $26^3 10^3$ passwords, since we have 26 choices for each letter, and 10 choices for each digit. The number of patterns is $\binom{6}{3}$, since we pick 3 positions out of 6 for the letters. Thus, the total number of passwords is $\binom{6}{3}26^3 10^3$.