University of Washington — February 6, 1998
Department of Computer Science and Engineering
CSE 321, Winter 1998
Handout No. 1

Very old midterm questions

Note: The previous midterms may have covered sections that you are not responsible for on the W98 Midterm.

Instructions: Closed books, closed notes, no calculators, work all problems. The problems all have equal weight.

Time Limit: 50 minutes.

Problem 1:
Let $T(n)$ be defined by:

$$T(0) = 1, \qquad T(n) = 2 * n * T(n-1).$$

Prove that for all $n \geq 0$, $T(n) = 2^n n!$.

Problem 2:
Let $x_1, x_2, \ldots, x_n$ be odd integers. Prove that $x_1 x_2 \cdots x_n$ is also an odd integer.

Problem 3:
Prove

$$\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}.$$

(Hint - you do not need to use induction.)

Problem 4:
Consider permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

a) How many permutations start with the number 1?

b) How many permutations have all of the even numbers appearing before any of the odd numbers?

(Answers should be expressed in factorial notation - do not multiply out your answer.)

Problem 5:
Determine whether the following compound proposition is a tautology, a contradiction, or a contingency:

$$((s \vee p) \wedge (s \vee \neg p)) \rightarrow ((p \rightarrow q) \rightarrow r)$$

Problem 6:

Find predicates $P(x)$ and $Q(x)$ such that $\forall x(P(x) \oplus Q(x))$ is true, but $\forall x P(x) \oplus \forall x Q(x)$ is false. ($\oplus$ denotes the exclusive-or).

Problem 7:

Consider the following game that is offered at Anderson's Casino:
Initially, you start out with \$1. A coin is flipped, if it is heads you double your money and play again, if it is tails the game stops, and you keep your money.
You know that the Casino is a somewhat sleazy place, infact, you know that the probability that the coin comes up heads is $\frac{1}{3}$, and the probability it comes up tails is $\frac{2}{3}$.

a) Give a formula that can be used to compute your expected winnings when you play this game.

b) Evaluate your formula to determine your expected winnings.

c) The Casino charges \$3 to play this game (although it does provide you with the initial \$1). Do you play?

Problem 1:

Suppose that a department offers 6 courses, and every student is required to take exactly 3 of the courses. Prove that if there are 25 students, then at least one pair of students takes exactly the same set of courses.
If the requirement is changed so that students take at least 3 courses, is there necessarily a pair of students taking the same set of courses? (Justify your answer).

Problem 2:

Let $T(n)$ be defined by:

$$T(0) = 1, \qquad T(1) = 2, \qquad T(n) = T(n-1) + 2T(n-2) \quad \text{for } n \geq 2.$$

Prove that for all $n \geq 0$, $T(n) = 2^n$.

Problem 3:

Prove that the sum of an odd number and an even number is an odd number.

Problem 4:

Let S be the set of length 10 over the alphabet $\{0, 1\}$.

a) How many strings in S have the same number of 1's and 0's.

b) How many strings in S start with 4 consecutive 1's and end with 4 consecutive 0's.

Problem 5:

Use mathematical induction to show that 3 divides $n^3 - n$ whenever n is a non-negative integer.

Problem 6:

Show that the following compound proposition is a contingency (i.e., neither a tautology nor a contradiction):

$$(p \to (q \to r)) \to (\neg p \to (\neg q \to \neg r))$$

Problem 7:

Find predicates $P(x)$ and $Q(x)$ such that $\forall x(P(x) \to Q(x))$ is false, but $\forall x P(x) \to \forall x Q(x)$ is true.

Problem 1:

Show that the following compound proposition is a contingency (i.e., neither a tautology nor a contradiction):

$$(p \to (q \oplus r)) \to r)$$

Problem 2:

Give predicates $P(x)$ and $Q(x)$ which show that $\exists x(P(x) \to Q(x))$ is not logically equivalent to $\exists x P(x) \to \exists x Q(x)$.

Problem 3:

Let the predicated $D(x, y)$ mean "team x defeated team y" and $P(x, y)$ mean "team x has played team y." Give quantified formulas with the following meanings:

a) Every team has lost at least one game.

b) There is a team that has beaten every team it has played.

Problem 4:

Prove that there exists an $n \geq 0$ such that $2^n + 1$ is not prime.

Problem 5:

Let g_n be defined by:

$$g_0 = 1, \qquad g_{n+1} = \sum_{i=0}^{n} g_i \quad \text{for } n \geq 0.$$

Prove that for $n \geq 1$, $g_n = 2^{n-1}$.

Problem 6:

Let the relation $R = \{\langle x, x+1 \rangle\}$ (in other words, xRy iff $x + 1 = y$). Prove that $R^n = \{\langle x, x+n \rangle\}$. Recall that R_n is defined recursively by $R_1 = R$ and $R^{n+1} = R^n \circ R$.