Natural Deduction
Notes for CSE 321 – Winter 2010
(Updated)

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Natural Deduction is the formal proof system that we will use in class. It consists of a set of rules that allow us to write *deductions*, or *proofs*. By applying these rules, and only these rules, one can prove any tautology in propositional calculus or in relational calculus. The proof starts from a set of *hypotheses*, Γ, uses several intermediate steps, then ends in a *conclusion*, which is a proposition (or predicate) p. There are two ways to denote a proof (or deduction). Top down: Γ at the top, then all intermediate steps, then p:

$$\begin{array}{c}\Gamma \\ \vdots \\ p\end{array}$$

or using the *turnstile*:

$$\Gamma \vdash p$$

Start by reading and understanding the top-down notation, then move on to the turnstile notation. In the homework and exams use turnstile.

To prove that p is a tautology we must find a derivation of $\vdash p$: that is p is true without any hypothesis.

Operators: $\texttt{T}$, $\texttt{F}$, $\wedge$, $\vee$, $\rightarrow$. There is no $\neg$: instead $\neg p$ is written as $p \rightarrow \texttt{F}$.

1 Natural Deduction Rules

The nice feature in *natural deduction* is that it has exactly two rules for each operator: an introduction and an elimination rule. There are two exceptions: no elimination for `T` and no introduction for `F`.

1.1 Intuitionistic Logic

Trivial Rule:

$$\frac{\Gamma, p}{p}$$

TRUTH-introduction Rule:

$$\frac{\Gamma}{\mathtt{T}}$$

FALSEHOOD-elimination Rule:

$$\frac{\mathtt{F}}{p}$$

AND-introduction Rule:

$$\frac{\begin{array}{cc}\Gamma & \Gamma\\ \vdots & \vdots\\ p & q\end{array}}{p \wedge q}$$

AND-elimination Rules:

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ p \wedge q\end{array}}{p} \qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ p \wedge q\end{array}}{q}$$

OR-introduction Rules:

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ p\end{array}}{p \vee q} \qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ q\end{array}}{p \vee q}$$

OR-elimination Rule:

$$\frac{\begin{array}{ccc}\Gamma & \bar{p} & \bar{q}\\ \vdots & \vdots & \vdots \\ p \vee q & r & r\end{array}}{r}$$

In English: "if we can (a) deduce $p \vee q$ from Γ, (b) deduce r from Γ, p, and (c) deduce r from Γ, q, then we can (d) deduce r from Γ". Notice that p and q are no longer premises for r; they may be removed from the hypothesis. This is represented by the bar above them; sometimes we just strike them out.

IMPLICATION-introduction Rule:

$$\frac{\begin{array}{c}\Gamma, \bar{p}\\ \vdots \\ q\end{array}}{p \rightarrow q}$$

In English: "if we can deduce q from Γ, p, then we can deduce $p \rightarrow q$ from Γ". Note that p may be removed from the hypothesis.

IMPLICATION-elimination Rule: This rule is also called **Modus ponens**.

$$\frac{\begin{array}{cc}\Gamma & \Gamma\\ \vdots & \vdots \\ p & p \rightarrow q\end{array}}{q}$$

For predicate calculus, we need two more rules:

UNIVERSAL-introduction

$$\frac{\begin{array}{c}\Gamma\\ \vdots \\ P\end{array}}{\forall x.P}$$

provided that x is not a free in any predicate in Γ.

In English: "if we can deduce P from Γ *without using any hypothesis containing* x, then we can deduce $\forall x.P$ from Γ".

UNIVERSAL-elimination

$$\frac{\forall x.P}{P[t/x]}$$

EXISTS-introduction Rule:

$$\frac{P[t/x]}{\exists x.P}$$

EXISTS-elimination Rule:

$$\frac{\begin{matrix}\Gamma & \overline{P} \\ \vdots & \vdots \\ \exists x.P & Q\end{matrix}}{Q}$$

where x does not occur free in any predicate in Γ, nor in Q.

In English: "if we can deduce $\exists x.P$ from Γ, and we can deduce Q from $P[t/x]$ (this denotes P where x is substituted with some term t), then we can deduce Q from Γ". Note that we need to remove $\exists.P$ from the hypothesis.

1.2 Classical Logic

Add any one of the following three rules to obtain *classical logic*. Recall that $\neg p$ means $p \rightarrow \mathtt{F}$.

Proof-by-contradiction Rule:

$$\frac{\begin{matrix}\Gamma, \overline{\neg p} \\ \vdots \\ \mathtt{F}\end{matrix}}{p}$$

In English: "if we an deduce $\mathtt{F}$ from $\Gamma, \neg p$, then we can deduce p from Γ".

Double negation Rule:

$$\frac{\Gamma}{\neg\neg p \rightarrow p}$$

Excluded middle Rule:

$$\frac{\Gamma}{p \vee \neg p}$$

2 Turnstile Notation

Rule name	Rule	
Trivial	$p \vdash p$	
TRUTH-introduction	$\vdash \mathtt{T}$	
FALSEHOOD-elimination	$\mathtt{F} \vdash p$	
AND-introduction	$p, q \vdash p \land q$	
AND-elimination	$p \land q \vdash p$	$p \land q \vdash q$
OR-introduction	$p \vdash p \lor q$	$q \vdash p \lor q$
OR-elimination	$\frac{\Gamma \vdash p \lor q;\ \Gamma, p \vdash r;\ \Gamma, q \vdash r}{\Gamma \vdash r}$	
IMPLICATION-introduction	$\frac{\Gamma, p \vdash q}{\Gamma \vdash p \rightarrow q}$	
IMPLICATION-elimination	$p, (p \rightarrow q) \vdash q$	
Proof by contradiction	$\frac{\Gamma, \neg p \vdash \mathtt{F}}{\Gamma \vdash p}$	
UNIVERSAL-introduction	$\frac{\Gamma \vdash P; x \notin \mathtt{free-variables}(\Gamma)}{\Gamma \vdash \forall x P}$	
UNIVERSAL-elimination	$\forall x.P \vdash P[t/x]$	
EXISTENTIAL-introduction	$\Gamma, P[t/x] \vdash \exists x.P$	
EXISTENTIAL-elimination	$\frac{\Gamma \vdash \exists x.P; \Gamma, P \vdash Q; x \notin \mathtt{free-variables}(\Gamma, Q)}{\Gamma \vdash Q}$	
CUT	$\frac{\Gamma \vdash p; \quad \Delta, p \vdash q}{\Gamma, \Delta \vdash q}$	

In CUT, the notation Γ, Δ means the union of all premises in Γ and in Δ. The CUT allows us to compose two deductions to produce a longer one. We will combine multiple cuts into one rule:

$$\frac{\Gamma_1 \vdash p_1; \ldots; \Gamma_n \vdash p_n; \quad \Delta, p_1, \ldots, p_n \vdash q}{\Gamma_1, \ldots, \Gamma_n, \Delta \vdash q}$$

Convince yourself that, if you were asked you use only the single-cut rule, you could always replace an instance of the multiple cut rule by using the single cut rule several times.

3 Examples

Example 3.1 Prove that $p \land q \rightarrow q \land p$.

The last rule that we want to use is IMPLICATION-introduction:

$$\dfrac{\begin{array}{c}\overline{p\wedge q}\\ \vdots\\ q\wedge p\end{array}}{p\wedge q\rightarrow q\wedge p}$$

Our goal now is to prove $q \wedge p$ from $p \wedge q$. Note that we do not use commutativity here: only the inference rules in natural deduction may be used. To prove it, we ask the question: what is the last rule that we need to prove $q \wedge p$? Obviously, AND-introduction:

$$\dfrac{\begin{array}{cc}p\wedge q & p\wedge q\\ \vdots & \vdots\\ q & p\end{array}}{q\wedge p}$$

It remains to prove q from $p \wedge q$ and p from $p \wedge q$. Both are a single application of the AND-elimination rule. Thus, the entire deduction is:

$$\dfrac{\frac{\frac{\overline{p\wedge q}\;\overline{p\wedge q}}{q\quad p}}{q\wedge p}}{p\wedge q\rightarrow q\wedge p}$$

Same in trunstile notation:

$$\dfrac{\frac{p\wedge q\vdash q;\quad p\wedge q\vdash p}{p\wedge q\vdash q\wedge p}}{\vdash p\wedge q\rightarrow q\wedge p}$$

Example 3.2 Prove that $p \vee (p \wedge q) \rightarrow p$.

What is the last rule that we want to apply ? IMPLICATION-introduction, hence we want to have this as the last step:

$$\dfrac{\begin{array}{c}\overline{p\vee(p\wedge q)}\\ \vdots\\ p\end{array}}{p\vee(p\wedge q)\rightarrow p}$$

So the goal is to prove p from the hypothesis $p \vee (p \wedge q)$. Here, the last rule we want to apply is OR-elimination:

$$\frac{p \vee (p \wedge q) \quad \begin{array}{c}\overline{p}\\ \vdots \\ p\end{array} \quad \begin{array}{c}\overline{p \wedge q}\\ \vdots \\ p\end{array}}{p}$$

Thus, we need to fill in the inner-most deductions. Both can be obtained by a single rule: trivial rule, and AND-elimination. Thus, the final deduction is:

$$\frac{\frac{\overline{p \vee (p \wedge q)} \quad \frac{\overline{p}}{p} \quad \frac{\overline{p \wedge q}}{p}}{p}}{p \vee (p \wedge q) \to p}$$

In turnstile notation:

$$\frac{\frac{p \vdash p; \quad p \wedge q \vdash p}{p \vee (p \wedge q) \vdash p}}{\vdash p \vee (p \wedge q) \to p}$$