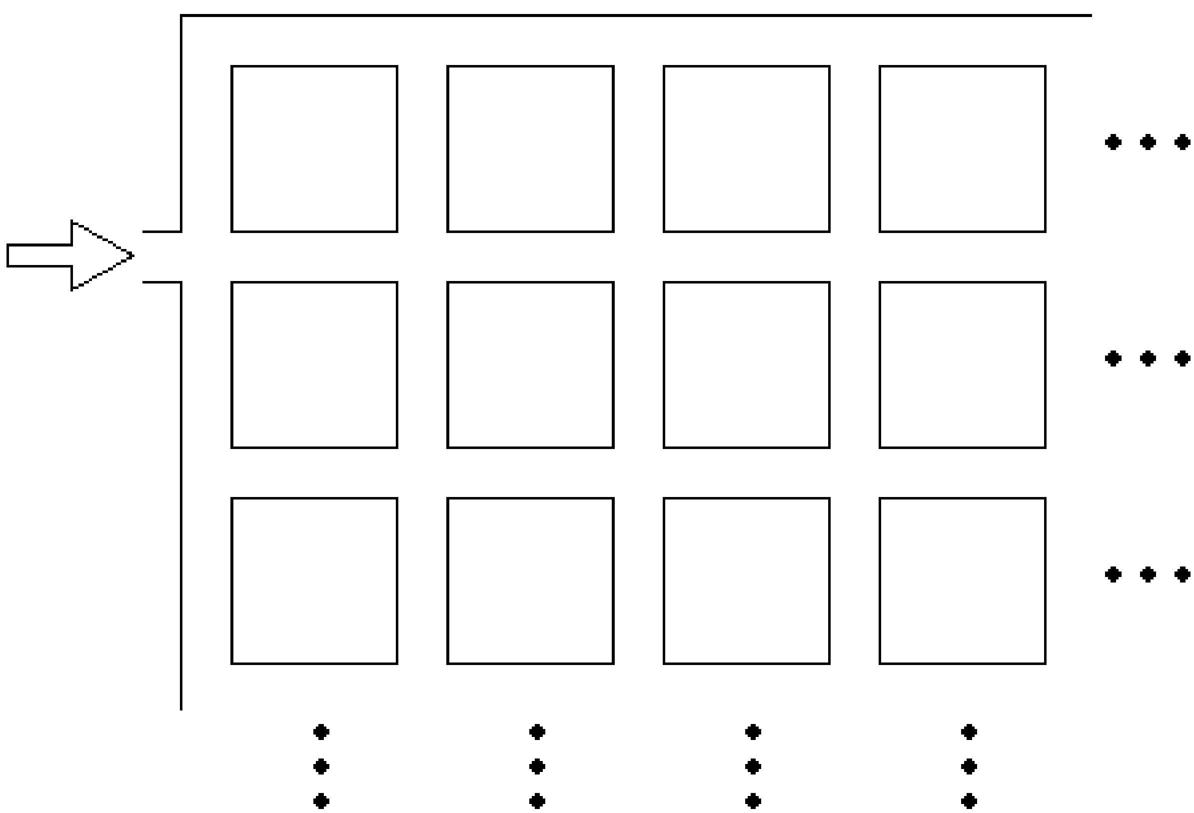
CSE 321 Winter 2007 Final Exam

1. (8 points) Sam Slacker usually oversleeps and misses his bus. Sometimes luck is on his side and the bus is running late, but even then he might still miss the bus. If the bus is running late, he catches it 70% of the time. If the bus is running on time, he catches it only 20% of the time. Metro Bus runs a fairly tight ship and busses run late only about 30% of the time. What is the probability that the bus was running late given that Sam managed to catch the bus?

2. (8 points) Show that for an any positive integer n, $4n+3$ and $5n+4$ are relatively prime. (Hint: Use Euclid's algorithm. Do not use induction.)

3. (8 points) The gated town of Squaresville is made up of 10 by 10 square blocks (see partial picture below). Sam Slacker is the new mailman for Squaresville. Everyday, he enters Squaresville at their only entrance (see arrow below) and delivers mail to all the residents.

   Being a slacker, Sam wants to do as little work as possible; he is looking for a way to reach every house without having to double back. Assume that once Sam passes a house (on either side of the street), then he can deliver mail to that house. Can Sam deliver mail to everyone without doubling back or retracing any of his steps? Why or why not?



4. (8 points)

   (a) "Mega Millions is the multi-state lottery game that has the biggest jackpot around. ... Just select five numbers from a field of 56, and any one number from a field of 46." The five numbers are unique and their order does not matter. The sixth number is not necessarily different from any of the first five numbers. How many ways are there to select six numbers to play Mega Millions?

   (b) If the rules were changed so that the five numbers did not have to be unique, how many ways would there be to select six numbers to play Mega Millions?

5. (8 points) Prove that $(q \wedge (p \rightarrow \neg q)) \rightarrow \neg p$ is a tautology without using a truth table.

6. (8 points) A palindrome is a string whose reversal is identical to the string. For strings over an alphabet of 26 letters (*e.g.*, the Roman alphabet), "kayak" and "qrexxerq" are examples of palindromes. Notice that the string does not necessarily have to be an English word. How many such strings of length n are palindromes? (Hint: Do not try to come up with a single formula for all n.)

7. (8 points)

   (a) Consider the following relation over the set of sets.

   $$R = \{(A, B) \mid A \cap B \neq \emptyset\}$$

   Is R an equivalence relation? Why or why not?

   (b) If so, let $A = \{1, 2, 3\}$. What is another set in the equivalence class of A? If not, leave this space blank.

8. (12 points)

   (a) A pair of cubical (standard) dice are rolled together. I'll pay you \$4 if the sum of the numbers are 10 or higher, otherwise you'll pay me \$1. What is your expected payoff?

   (b) An *octahedral* die has eight (8) faces that are numbered 1 through 8. A *dodecahedral* die has twelve (12) faces that are numbered 1 through 12. What is the expected value of the sum of the numbers that come up when a fair octahedral die and a fair dodecahedral die are rolled together?

9. (8 points) Prove the statement: "For any integers a, b, and c, if there exists an integer m such that $b \not\equiv c \pmod{m}$, then either $a \neq b$ or $a \neq c$."

10. (8 points)

    (a) Consider a relation R over the set $S = \{a, b, c, d, e\}$. How many such relations are there? Explain. Try not to use more than 50 words.

    (b) Suppose you chose one such relation uniformly at random. What is the probability that the size of the relation is at least 3?

11. (8 points) Prove that at a party where there are at least two people, there are two people who know the same *number* of other people there.

12. (8 points) Prove that for every positive integer n,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Equivalences	
Identity Laws	$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
Domination Laws	$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Negation Laws	$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$
Double Negation Law	$\neg\neg p \equiv p$
Contrapositive Law	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
Implication Law	$p \rightarrow q \equiv \neg p \vee q$
Quantifier Negation Laws	$\neg\exists x P(x) \equiv \forall x \neg P(x)$ $\neg\forall x P(x) \equiv \exists x \neg P(x)$

Propositional and Predicate Equivalences

Inferences	
Modus Ponens	$\dfrac{p,\ p \to q}{\therefore q}$
Direct Proof	$\dfrac{p \Rightarrow q}{\therefore p \to q}$
Simplification	$\dfrac{p \wedge q}{\therefore p,\ q}$
Consolidation	$\dfrac{p,\ q}{\therefore p \wedge q}$
Disjunctive Syllogism	$\dfrac{p \vee q,\ \neg p}{\therefore q}$
Addition	$\dfrac{p}{\therefore p \vee q,\ q \vee p}$
Excluded Middle	$\dfrac{}{\therefore p \vee \neg p}$
Universal Instantiation	$\dfrac{\forall x P(x)}{\therefore P(c):\ c \text{ arbitrary}}$
Universal Generalization	$\dfrac{P(c):\ c \text{ arbitrary; no dependency}}{\therefore \forall x P(x)}$
Existential Instantiation	$\dfrac{\exists x P(x)}{\therefore P(c):\ c \text{ new and specific; depends on ...}}$
Existential Generalization	$\dfrac{P(c):\ c \text{ specific or arbitrary}}{\therefore \exists x P(x)}$

Propositional and Predicate Inferences