CSE 321 Discrete Structures

March 5th and 8th, 2010

Lecture 23-24: Relations

Announcements

- Readings for Monday:
  - Section 8.1 Binary Relations
  - Section 8.2 n-Ary relations
  - Section 8.3 Representing Relations
  - Section 8.4 Closures of Relations
  - Section 8.5 Equivalence Relations
  - Section 8.6 Partial Orderings

Definition of Relations

Let A and B be sets,
A binary relation from A to B is a subset of $A \times B$

Let A be a set,
A binary relation on A is a subset of $A \times A$

Properties of Relations

Let R be a relation on A

R is reflexive iff $(a,a) \in R$ for every $a \in A$

R is symmetric iff $(a,b) \in R$ implies $(b, a) \in R$

R is antisymmetric iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \in R$

R is transitive iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

Combining Relations

Let R be a relation from A to B
Let S be a relation from B to C
The composite of R and S, $S \circ R$ is the relation from A to C defined by:

$$S \circ R = \{(a, c) \mid \exists\ b \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$$

Examples

(a,b)$\in$ Parent: b is a parent of a

(a,b)$\in$ Sister: b is a sister of a

What is Parent ° Sister?

What is Sister ° Parent?

$S \circ R = \{(a, c) \mid \exists\, b \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$

Examples

Using the relations: Parent, Child, Brother, Sister, Sibling, Father, Mother express

Uncle: b is an uncle of a

Cousin: b is a cousin of a

Powers and Transitivity

$R^2 = R \circ R = \{(a, c) \mid \exists\, b \text{ such that } (a,b) \in R \text{ and } (b,c) \in R\}$

$R^0 = \{(a,a) \mid a \in A\}$

$R^1 = R$

$R^{n+1} = R^n \circ R$

R is Transitive if and only if $R^n \subseteq R$ for all $n \geq 1$

n-ary relations

Let A_1, A_2, …, A_n be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

Relational databases

Student_Name	ID_Number	Major	GPA
Knuth	328012098	CS	4.00
Von Neuman	481080220	CS	3.78
Von Neuman	481080220	Mathematics	3.78
Russell	238082388	Philosophy	3.85
Einstein	238001920	Physics	2.11
Newton	1727017	Mathematics	3.61
Karp	348882811	CS	3.98
Newton	1727017	Physics	3.61
Bernoulli	2921938	Mathematics	3.21
Bernoulli	2921939	Mathematics	3.54

Alternate Approach

Student_ID	Name	GPA
328012098	Knuth	4.00
481080220	Von Neuman	3.78
238082388	Russell	3.85
238001920	Einstein	2.11
1727017	Newton	3.61
348882811	Karp	3.98
2921938	Bernoulli	3.21
2921939	Bernoulli	3.54

Student_ID	Major
328012098	CS
481080220	CS
481080220	Mathematics
238082388	Philosophy
238001920	Physics
1727017	Mathematics
348882811	CS
1727017	Physics
2921938	Mathematics
2921939	Mathematics

Database Operations

Projection

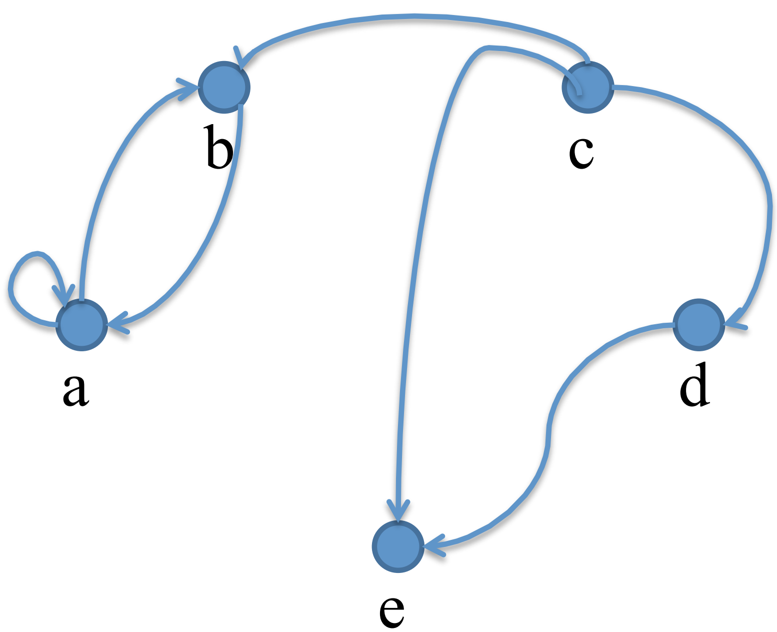
Join

Select

Representation of relations

Directed Graph Representation (Digraph)

{(a, b), (a, a), (b, a), (c, a), (c, d), (c, e) (d, e) }



Matrix representation

Relation R from A={a_1, … a_p} to B={b_1, . . . b_q}

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

{(1, 1), (1, 2), (1, 4), (2,1), (2,3), (3,2), (3, 3) }

Matrix operations

How do you tell if a relation is reflexive from its adjacency matrix?

How do you tell if a relation is symmetric from its adjacency matrix?

Suppose R has matrix M_R and S has Matrix M_S.
What are the matrices for $R \cup S$ and $R \cap S$?

Matrix multiplication

Standard (×, +) matrix multiplication.
A is a m × n matrix, B is a n × p matrix
C = A × B is a m × p matrix defined:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

And-OR Matrix multiplication

A is a m × n boolean matrix, B is a n × p boolean matrix
C = A ⊗ B is a m × p matrix defined:

$$c_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} (a_{11} \wedge b_{11}) \vee (a_{12} \wedge b_{21}) \vee (a_{13} \wedge b_{31}) & (a_{11} \wedge b_{12}) \vee (a_{12} \wedge b_{22}) \vee (a_{13} \wedge b_{32}) & \cdots \\ (a_{21} \wedge b_{11}) \vee (a_{22} \wedge b_{21}) \vee (a_{23} \wedge b_{31}) & (a_{21} \wedge b_{12}) \vee (a_{22} \wedge b_{22}) \vee (a_{23} \wedge b_{32}) & \cdots \\ (a_{31} \wedge b_{11}) \vee (a_{32} \wedge b_{21}) \vee (a_{33} \wedge b_{31}) & (a_{31} \wedge b_{12}) \vee (a_{32} \wedge b_{22}) \vee (a_{33} \wedge b_{32}) & \cdots \end{bmatrix}$$

Matrices and Composition

$$M_{S\circ R} = M_R \otimes M_S$$

R = {(a, a), (a, c), (b, a), (b, b)}
S = {(b, a), (b, c), (c, a), (c, c)}

Closures

- Reflexive Closure
- Symmetric Closure
- Transitive closure
  - $R = \{(1, 2), (2, 3), (3, 4)\}$

Equivalence Relations

Definition: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Are these equivalence relations?

- Congruence Mod m on $\mathbf{Z}^+$. R = {(a,b) | a $\equiv$ b mod m}

- The 'divides' relation on $\mathbf{Z}^+$. R = {(a,b) | a|b}

Equivalence classes

- $R = \{(a,b) \mid a \equiv b \bmod 3\}$, Domain: $\mathbf{Z}^+$

Partial Orderings

Definition: A relation R on a set S is called a *partial ordering* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*.

Are these posets?

- $(\mathbf{Z}, \geq)$
- $(\mathbf{Z}^{+}, |)$

Total Orderings

Definition: If (S, R) is a poset and every two elements of S are comparable, S is called a *totally (linearly) ordered set*, and R is called a *total (linear) order*.

Are these posets totally ordered?

- $(\mathbf{Z}, \leq)$
- $(\mathbf{Z}^+, |)$

Ordering examples

- Total Orders
  - Lexicographic Order
- Partial Orders
  - Prerequisites
  - Dominance order