CSE 321 Discrete Structures

January 20, 2010

Lecture 07: Induction

Announcements

- Reading from the textbook: Chapter 4

- Homework 1 is graded: check grades here https://catalysttools.washington.edu/gradebook/ahhunter/17763

- Homework 2
  - Due date: Friday, Jan 22

Outline

- Mathematical induction
- Strong induction
- Inductive definitions
- Structural induction

Induction Example

- Prove $3 \mid 2^{2n} - 1$ for $n \geq 0$

Induction as a rule of Inference

$$\frac{P(0) \quad ; \quad \forall k.(P(k) \rightarrow P(k+1))}{\forall n.P(n)}$$

Sums

$$1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$$

Prove this by induction

Sums

$$f(n) = 1 + 2 + \ldots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Prove by induction

$$f(n) = 1 + 3 + 5 + \ldots + (2n-1) = \sum_{i=1}^{n} (2i-1) = n^2$$

$$f(n) = 1^2 + 2^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$f(n) = 1^3 + 2^3 + \ldots + n^3 = \ ?$$

Find the sums, then prove by induction

$$f(n) = 1^4 + 5^4 + 9^4 + \ldots + (4n-3)^4 = \sum_{i=1}^{n} (4i-3)^4 = \ ?$$

Harmonic Numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

Prove $H_{2^n} \geq 1 + \frac{n}{2}$

More Sums

$$f(n) = 1 + 3 + 3^2 + \ldots + 3^n = \sum_{i=1}^{n} 3^i = \frac{3^{n+1} - 1}{3 - 1}$$

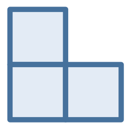
Sometimes sums are easiest computed with integrals:

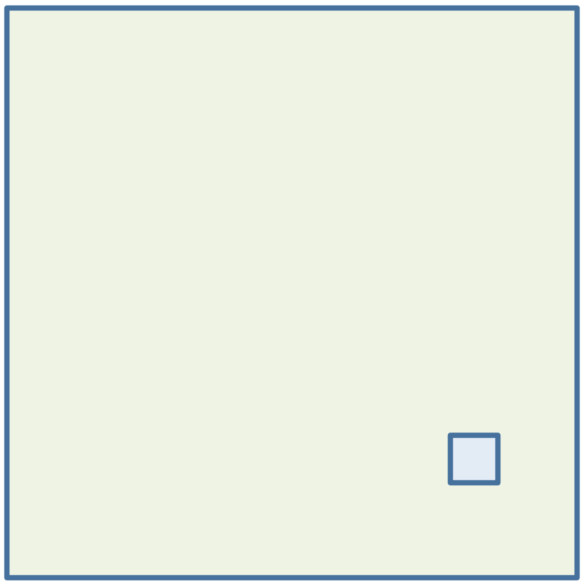
$$f(n) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i} \approx 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln(n) - \ln(1)$$

$$f(n) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} = \sum_{i=1}^{n} \frac{1}{i^2} \approx 1 + \int_1^n \frac{1}{x^2}\,dx = 1 + \frac{1}{1} - \frac{1}{n}$$

Using these hints, find upper/lower bounds, then prove them by induction

Cute Application: Checkerboard Tiling with Trinominos

Prove that a $2^k \times 2^k$ checkerboard with one square removed can be tiled with: 



Strong Induction

$$\frac{P(0); \quad \forall k.((P(0) \land P(1) \land P(2) \land \ldots \land P(k)) \rightarrow P(k+1))}{\forall n\ P(n)}$$

Better:

$$\frac{P(0); \qquad \forall k.\ (\ (\forall i \leq k.P(i)) \ \rightarrow \ P(k+1))}{\forall n\ P(n)}$$

Strong Induction Example

- Construct the following sequence:

$$a_0 = 1$$
$$a_{n+1} = a_0 + a_1 + \ldots + a_n$$

- Prove that: $\forall k \geq 1,\ a_k = 2^{k-1}$

Strong Induction Example

- Let P(k) be the statement: $a_k = 2^{k-1}$
- We prove P(k) by strong induction on k
- P(1): $a_1 = a_0 = 1$ and $2^0 = 1$; they are equal.
- Assume $k \geq 1$, and $\forall i \leq k$, P(i) is true: that is, $a_i = 2^{i-1}$. Then:

$$
\begin{aligned}
a_{k+1} &= a_0 + a_1 + \ldots + a_k = \\
&= 1 + (1 + 2 + 2^2 + \ldots + 2^{k-1}) \\
&= 1 + 2^k - 1 \\
&= 2^k
\end{aligned}
$$

Induction Example

- A set of S integers is *non-divisible* if there is no pair of integers a, b in S where a divides b. If there is a pair of integers a, b in S, where a divides b, then S is *divisible.*
- Given a set S of n+1 positive integers, none exceeding 2n, show that S is divisible.
- What is the largest subset non-divisible subset of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }.

If S is a set of n+1 positive integers, none exceeding 2n, then S is divisible

- Base case: n =1

- Suppose the result holds for n
  - If S is a set of n+1 positive integers, none exceeding 2n, then S is divisible
  - Let T be a set of n+2 positive integers, none exceeding 2n+2.

Proof by contradiction

Suppose T is non-divisible.

- Claim: $2n+1 \in T$ and $2n + 2 \in T$
- Claim: $n+1 \notin T$
- Let $T^* = T - \{2n+1, 2n+2\} \cup \{n+1\}$
- If T is non-divisible, T* is also non-divisible

The Game Of Nim

- Several Matches are placed in rows
- Player 1 removes any number of matches from some row
- Player 2 removes any number of matches from some row
- Last player to remove a match wins

Row 1: |
Row 2: | | |
Row 3: | | | | | |

The Game Of Nim

- Prove that in the game with two rows and equal number of matches, the second player has a winning strategy

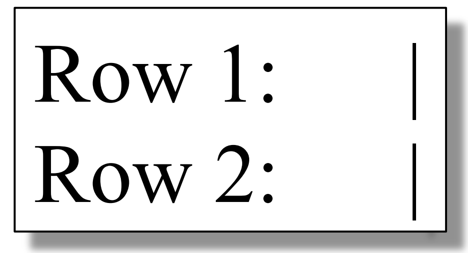
Row 1: | | | | | | | | | | | |
Row 2: | | | | | | | | | | | |

The Game Of Nim

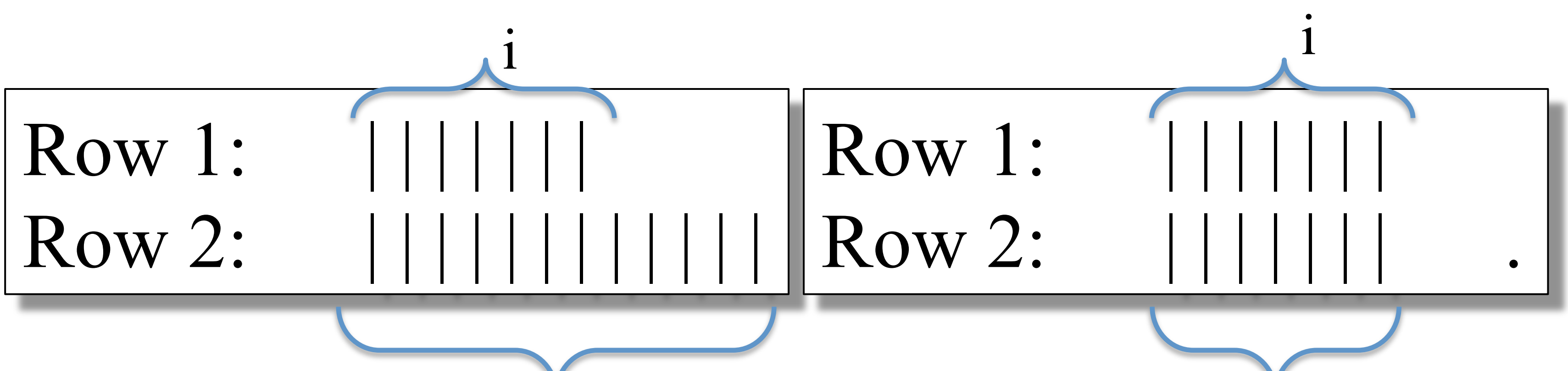
- Let P(k) be the statement: “Player 2 has a winning strategy in a game of Nim with two rows, where each row has k matches”.
- We prove P(k) by induction on k

The Game Of Nim

- P(1): player 1 must remove one match; player 2 wins



- Assume k ≥ 1, and ∀i≤k, P(i) is true
  Suppose player 1 removes some matches from the first row, and leaves i matches. Then player 2 removes the same numberf of matches from row 2. Now wse use the fact that P(i) is true:



Recursive Definitions

- $F(0) = 0$; $F(n + 1) = F(n) + 1$;

- $F(0) = 1$; $F(n + 1) = 2 \times F(n)$;

- $F(0) = 1$; $F(n + 1) = 2^{F(n)}$

Fibonacci Numbers

- $f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$

Bounding the Fibonacci Numbers

- Theorem: $2^{n/2} \leq f_n \leq 2^n$ for $n \geq 6$

More Recursive Definitions

- $f(n) = 2f(n-1) + 1, \quad f(0) = T$
- Telescoping

← First, find the expression f

$$
\begin{aligned}
\Rightarrow f(n)+1 &= 2(f(n-1)+1) \\
f(n-1)+1 &= 2(f(n-2)+1) \quad \times 2 \\
f(n-2)+1 &= 2(f(n-3)+1) \quad \times 2^2 \\
&\ldots\ldots \\
f(1)+1 &= 2(f(0)+1) \quad \times 2^{n-1}
\end{aligned}
$$

$$\Rightarrow f(n)+1 = 2^n(f(0)+1) = 2^n(T+1)$$

$$\Rightarrow f(n) = 2^n(T+1) - 1$$

← Next, prove this by induction

More Recursive Definitions

- Fibonacci: f(n) = f(n-1)+f(n-2), f(0)=f(1)=1

← First, find the expression f

➔ try $f(n) = A c^n$ What is c ?

$A c^n = A c^{n-1} + A c^{n-2}$, $c^2 - c - 1 = 0$

$$c_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$f(n) = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

← Next, prove this by induction

Recursive Definitions of Sets

- Recursive definition
  - Basis step: $0 \in S$
  - Recursive step: if $x \in S$, then $x + 2 \in S$
  - Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

Recursive definitions of sets

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$, then $x + y \in S$;

What is this set ?

Strings

- The set Σ^* of strings over the alphabet Σ is defined
  - Basis: $\lambda \in S$ (λ is the empty string)
  - Recursive: if $w \in \Sigma^*$, $x \in \Sigma$, then $wx \in \Sigma^*$

Function definitions

$\text{Len}(\lambda) = 0;$
$\text{Len}(wx) = 1 + \text{Len}(w);$ for $w \in \Sigma^*$, $x \in \Sigma$

$\text{Concat}(w, \lambda) = w$ for $w \in \Sigma^*$
$\text{Concat}(w_1, w_2 x) = \text{Concat}(w_1, w_2)x$ for w_1, w_2 in Σ^*, $x \in \Sigma$

Using Induction for Program Correctness

- Mystery program: what does it compute ?

```
public class mystery
{
   public static void main( String [ ] args )
   {
      int a = . . . . ; int b = . . .;
      int x = a; int y = b; int z = 0;
          while (x > 0) {
             if ((x & 1) == 0) { x >>= 1; y <<= 1; }
             { x--; z += y; }
          }
          /* what does this program compute from a and b ? */
   }
}
```