CSE 321  Discrete Structures

Winter 2008
Lecture 17
Counting

Announcements

• Readings
  – Counting
    • 5.3, (4.3) Permutations and Combinations
    • 5.4, (4.4) Binomial Coefficients
    • 5.5, (4.5) Generalized Permutations and Combinations
  – Homework

Highlights from Lecture 16

• Counting
  – Product Rule
    • $|A_1 \times A_2| = |A_1||A_2|$  
  – Sum Rule
    • $A_1$ and $A_2$ disjoint, $|A_1 \cup A_2| = |A_1| + |A_2|$
  – Inclusion-Exclusion
    • $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
• Pigeon Hole Principle

Clever PHP Applications

• Every sequence of $n^2 + 1$ distinct numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

4, 22, 8, 15, 19, 11, 2, 1, 9, 20, 10, 7, 16, 3, 6, 5, 14

Proof

• Let $a_1, \ldots, a_n$ be a sequence of $n^2+1$ distinct numbers  
• Let $i_k$ be the length of the longest increasing sequence starting at $a_k$  
• Let $d_k$ be the length of the longest decreasing sequence starting at $a_k$  
  Suppose $i_k \leq n$ and $d_k \leq n$ for all $k$  
  There must be $k$ and $j$, $k < j$, with $i_k = i_j$ and $d_k = d_j$

Permutations vs. Combinations

• How many ways are there of selecting 1st, 2nd, and 3rd place from a group of 10 sprinters?

• How many ways are there of selecting the top three finishers from a group of 10 sprinters?
r-Permutations

• An r-permutation is an ordered selection of r elements from a set
• \( P(n, r) \), number of r-permutations of an n element set

\[
P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = \frac{n!}{(n - r)!}
\]

r-Combinations

• An r-combination is an unordered selection of r elements from a set (or just a subset of size r)
• \( C(r, n) \), number of r-permutations of an n element set

\[
C(n, r) = \binom{n}{r} = \frac{n!}{(n - r)!r!}
\]

How many

• Binary strings of length 10 with 3 0’s
• Binary strings of length 10 with 7 1’s
• How many different ways of assigning 38 students to the 5 seats in the front of the class
• How many different ways of assigning 38 students to a table that seats 5 students

Prove \( C(n, r) = C(n, n-r) \) [Proof 1]

• Proof by formula

\[
\binom{n}{r} = \frac{n!}{r!(n - r)!}
\]

\[
\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}
\]

Prove \( C(n, r) = C(n, n-r) \) [Proof 2]

• Combinatorial proof

Counting paths

• How many paths are there of length \( n+m-2 \) from the upper left corner to the lower right corner of an \( n \times m \) grid?
Binomial Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\]

= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n

Binomial Coefficient Identities from the Binomial Theorem

\[\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x + y)^n\]

\[\sum_{k=0}^{n} \binom{n}{k} = 2^n\]

\[\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\]

\[\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n\]

Pascal's Identity and Triangle

\[\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\]

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(3)  (6)  (6)  (3)
(3)  (6)  (6)  (3)
(1)  (3)  (3)  (1)
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\[\binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10\]