CSE 321: Discrete Structures Spring 2007

PROBLEM SET 8

Due Friday, June 1, 2007, in class

There are SEVEN problems, each worth 10 points.

Reading Assignment: 6th Edition: 8.4,8.5, 9.1-9.4. 5th edition: 7.4,7.5, 8.1-8.4.

---

1. For the relation $R = \{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$ on $\{a, b, c, d, e\}$, draw the following relations in digraph form:
   (a) The reflexive closure of R.
   (b) The symmetric closure of R.
   (c) The transitive closure of R.
   (d) The reflexive, symmetric, transitive closure of R.
2. Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n.
3. Let p be an odd prime, and let $A = \{1, 2, \ldots, p-1\}$. Define a relation T on A by $T = \{(a, b) \in A \times A \mid a \equiv 2^s b \pmod{p}$ for some *non-negative* integer $s\}$. Prove that T is an equivalence relation.
4. A relation R is called *circular* if $a\ R\ b$ and $b\ R\ c$ imply that $c\ R\ a$. Show that R is reflexive and circular if and only if it is an equivalence relation.
5. Consider a tournament among n players where each player plays every other player exactly once and all the $\binom{n}{2}$ games thus played result in a win/loss result. Define a natural "defeat" relation D on the set of players as follows: $a\ D\ b$ if player a defeated player b in their head-to-head encounter. Prove that the relation D has the following property: there exists a player w such that for every other player x either $w\ D\ x$ or there exists a player z such that $w\ D\ z$ and $z\ D\ x$.
   (Hint: Try for the candidate w a player who has defeated the most other players.)
6. Prove that any simple, undirected graph on $n \geq 2$ vertices contains two vertices of equal degree.
7. Suppose G is a simple, undirected graph on $2n$ vertices that contains no triangles (cycles of length 3). Prove that G has at most n^2 edges. (Hint: Induction on n is one possible approach.)