CSE 321: Discrete Structures Spring 2007

PROBLEM SET 3

Due Friday, April 20, 2007, in class

Instructions: Same as for Problem Set 1.

The exercise numbers refer to the number in Rosen's book, 6th Edition. When a different number is used in the 5th edition, that number is also mentioned.

1. Prove that if you pick 10 distinct numbers from 1 to 1000, then there is a pair of numbers such that the larger of the two is less than twice the other.

2. Prove that if b and c are positive integers and bc is even, then b is even or c is even.

3. Prove or disprove: The value $2n^2 + 29$ is always prime for every integer $n > 0$.

4. (a) Prove that if n is an integer then $n^2 \mod 8$ is either 0,1, or 4.

   (b) Does the equation $3x^2 - 2y^2 = 85$ have any solutions where x and y are both integers? Why, or why not?

5. Prove that a positive integer n is divisible by 3 *if and only if* the sum of digits of n written in decimal is divisible by 3.

6. Sometimes it is easier to prove a stronger statement than is apparently required. In this problem, you will prove by mathematical induction that for all $n \geq 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 .$$

   (a) What doesn't work if one tries to produce an inductive proof in which $P(n)$ is the statement $\sum_{k=1}^{n} \frac{1}{k^2} < 2$?

   (b) Now use induction to prove the stronger statement that for all $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$.

7. What is wrong with this "proof"?

   "Theorem": For every even positive integer n, if x, y are positive integers such that $x+y = n$, then $x = y = n/2$.

   Basis step: Suppose that $n = 2$. If x, y are positive integers and $x + y = 2$, then we have $x = 1$ and $y = 1$.

   Inductive step: Let $n \geq 2$ be an even number. Assume that whenever $x + y = n$ and x, y are positive integers, $x = y = n/2$. Now suppose $x + y = n + 2$, where x, y are positive integers. Then $(x - 1) + (y - 1) = n$, and therefore by induction hypothesis, $x - 1 = y - 1 = n/2$. This implies that $x = y = (n + 2)/2$, completing the inductive step.

8. Section 4.2, Exercise 14. (5th Edition: Section 3.3, Exercise 40.)
   You are welcome to use any proof method, including mathematical induction. If you are thinking of arguments besides induction, let me say as a hint that the claimed bound follows from a slick one sentence proof.