CSE 321: Discrete Structures | Assignment #6
November 13, 2006 | Due: Monday, November 20

Reading Assignment: Rosen's text 6th Edition: finish sections 5.3-5.4 and read sections 6.1-6.4 (to page 433) or 5th Edition: sections 5.1-5.3 (to page 385).

Problems:

1. An ice cream parlor has 31 different flavors, 8 different kinds of sauce, and 12 toppings.
   (a) In how many different ways can a dish of three scoops of ice cream be made where each flavor can be used more than once and the order of the scoops does not matter?
   (b) How many different kinds of small sundaes are there if a small sundae contains one scoop of ice cream, a sauce, and a topping?
   (c) How many different kinds of large sundaes are there if a large sundae contains three scoops of ice cream, where each flavor can be used more than once and the order of the scoops does not matter; two kinds of sauce, where each sauce can be used only once and the order of the sauces does not matter; and three toppings, where each topping can be used only once and the order ot toppings does not matter?
2. How many ways are there to select 7 cards from a standard deck of 52 playing cards so that there are at least 5 cards of the same suit?
3. (a) 6th edition: Section 5.4, Exercise 8. 5th edition: Section 4.4, Exercise 8.
   (b) 6th edition: Section 5.4, Exercise 10. 5th edition: Section 4.4, Exercise 10.
4. 6th edition: Section 5.4, Exercise 24. 5th edition: Section 4.4, Exercise 24.
5. 6th edition: Section 5.4, Exercise 28. 5th edition: Section 4.4, Exercise 28.
6. 6th edition: Section 6.1, Exercise 28. 5th edition: Section 5.1, Exercise 28.
7. Which is more likely: rolling a total of exactly 8 when two dice are rolled or rolling a total of exactly 8 when three dice are rolled? Justify your answer.
8. A deck of 10 cards, each bearing a distinct number from 1 to 10, is shuffled to mix the cards thoroughly, so that each order is equally likely. What is the probability that the top three cards are in increasing order?
9. A fair coin is flipped n times. What is the probability that all the heads occur at the start of the sequence?

10. **Extra Credit (The Zen of counting):** One hundred ants are placed on a yard stick. (A yard stick is a ruler of length 1 yard = 36 inches = 91.44 centimeters.) Some are facing to the left, and some are facing to the right. When an ant reaches the end of the yard stick, it falls off. When two ants bump into each other, they turn around and head in opposite directions. Suppose that all ants travel at the same speed.

    - If the ants are placed so that their initial orientations are alternating—i.e. going from one end of the yard stick to the other, we see a leftward facing ant, then a rightward facing ant, then a leftward facing ant, etc.—how many total ant collisions are there before they all fall off the yard stick?

11. **Extra Credit: Due the week after Thanksgiving.** In cryptography, one typically needs to choose random primes of a certain size. In order to do this people simply choose random numbers and then check to see if they are prime. In order for this to work efficiently, the number of primes has to be plentiful. The following sequence of problems will direct you to produce a proof that primes are indeed plentiful. (The exact answer is closely related to the Reimann Hypothesis whose solution is worth a $1 million prize.)

    (a) Show that for any prime p, the largest power of p that divides $n!$ is

    $$\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \cdots + \lfloor \frac{n}{p^r} \rfloor$$

    where $p^r \leq n < p^{r+1}$.

    (b) Use the basic definition (no induction) to show that for any $m \geq 1$, $\lfloor \frac{2n}{m} \rfloor \leq 2\lfloor \frac{n}{m} \rfloor + 1$.

    (c) Use the formula for $\binom{2n}{n}$ and the results of parts (a) and (b) to show that for any prime p, the largest power p^r of p that divides $\binom{2n}{n}$ satisfies $p^r \leq 2n$.

    (d) Prove that for any integer $n \geq 1$, $\binom{2n}{n} \geq 2^n$.

    (e) Use the lower bound on the size of $\binom{2n}{n}$ from part (d) and upper bound on each of its prime power factors from part (c) to prove that the number of distinct primes dividing $\binom{2n}{n}$ is at least $n/\log_2(2n)$.

    (f) Conclude that there are at least $n/\log_2(2n)$ primes less than $2n$.

    NOTE: You can use the results of previous parts to solve later parts even if you haven't finished the earlier parts.