CSE 321
Practice Midterm

1. (18 points) Let $D(a, b)$ be the predicate "a divides b" (i.e., there is an integer c such that $b = ac$).

   (a) (8 points, 4 points each) Which of the following statements are true in the domain of positive integers?

   - $\forall x\ \forall y\ \forall z\ [D(x, y) \rightarrow D(x, yz)]$.

   - $\forall x\ \forall y\ \forall z\ \forall w\ [D(xy, zw) \rightarrow (D(x, z) \vee D(y, w))]$.

   (b) (10 points, 5 points each) Express the following statements as predicate logical formulas using the predicate D. (You can also use the predicates "$a > b$", "$a < b$", or "$a \neq b$" if you need them.)

   - "p is a prime number".

   - "q has at least two positive divisors greater than 1".

2. **True or false**: (52 points, 4 points each)

- $\neg p \vee (\neg q \wedge \neg r)$ is logically equivalent to $\neg((p \wedge q) \vee (p \wedge r))$.

- The negation of a tautology is a contradiction.

- $(\forall x P(x)) \wedge q$ is logically equivalent to $\forall x(P(x) \wedge q)$.

- If there is an onto function from a finite set A to a finite set B and there is an onto function from B to A, then there is a bijection from A to B.

- The proposition $p \rightarrow \neg(r \wedge s)$ is logically equivalent to the proposition $\neg p \vee \neg r \vee \neg s$.

- If $\exists x \forall y P(x, y)$ is true, then $\forall y \exists x P(x, y)$ is also true.

- $\forall x \forall y P(x, y)$ is logically equivalent to $\forall y \forall x P(x, y)$.

- If A, B and C are sets such that $A \cup C = B \cup C$, then $A = B$.

- If A, B and C are sets such that $A \cap C = B \cap C$, then $A = B$.

- The union of two countable sets is countable.

- If $f(x)$ and $g(x)$ are injective functions from A to B, and $h(x)$ is a function from A to B defined by $h(x) = f(x) + g(x)$, then h is also an injective function.

- If $a = bq + r$ where $0 \leq r < b$, then $gcd(a, b) = gcd(b, r)$.

- If $a \equiv r(\bmod b)$ and $0 \leq r < b$, then $gcd(a, b) = gcd(b, r)$.

3. (20 points) Prove by induction that

$$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots + n \cdot 2^{n-1} = (n-1) \cdot 2^n + 1$$

whenever n is a positive integer.

4. (12 points) Consider the following proof:

Claim: *Every natural number is either prime or a perfect square.*[1]

Proof: *We prove by induction that for all natural numbers n, $P(n)$: n is a prime or a perfect square.*
Base case: $P(1)$ *is certainly true.*
Inductive Hypothesis: *Every natural number less than n is a prime or a perfect square.*
Inductive step: *Consider n. If n is prime, then we are done. Otherwise, n can be factored as $n = rs$ with r and s less than or equal to $n-1$. By the inductive hypothesis, r and s are perfect squares, so $r = u^2$ and $s = v^2$. Therefore, $n = rs = u^2v^2 = (uv)^2$. So n is a perfect square.*
Therefore every natural number is either prime or a perfect square.

- (4 points) Does this proof use regular induction or strong induction?

- (8 points) Which of the following statements are true? (There could be zero, one, or more than one true statement.)
  (a) The proof is wrong because the inductive hypothesis is applied incorrectly. The inductive hypothesis asserts that r and s are **either** perfect squares **or** primes, but the proof uses it to conclude that r and s are perfect squares, ignoring the possibility that they are primes.
  (b) The proof is wrong because it proceeds by trying to prove that n is either a prime or a perfect square. But that is already the inductive hypothesis. Instead the proof should proceed by showing that $n + 1$ is either prime or a perfect square.
  (c) The proof is wrong because it is incorrect to claim that "If n is prime, then we are done," because this is what we were trying to prove.
  (d) The proof is correct.

[1] An integer a is a *perfect square* if there is an integer b such that $a = b^2$.