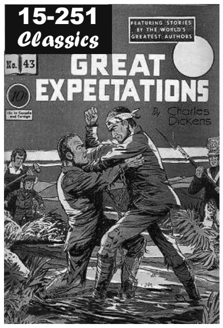
Great Theoretical Ideas In Computer Science

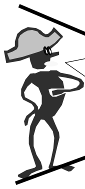
(Steven Rudich) John Lafferty — CS 15-251 — Fall 2005

Lecture 20 Nov. 3, 2005 — Carnegie Mellon University



Today, we will learn about a formidable tool in probability that will allow us to solve problems that seem <u>really really messy</u>…

If I randomly put 100 letters into 100 addressed envelopes, <u>on average</u> how many letters will end up in their correct envelopes?

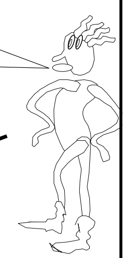


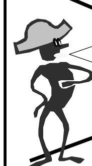
~~Hmm…~~

~~$\sum_k$ k·Pr(*exactly* k letters end up in correct envelopes)~~

~~= $\sum_k$ k· (…aargh!!…)~~

On average, in class of size m, how many pairs of people will have the same birthday?



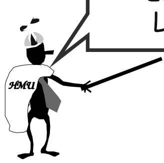


~~$\sum_k$ k· Pr(exactly k collisions)~~

~~= $\sum_k$ k· (…aargh!!!!…)~~

The new tool is called "Linearity of Expectation"

Expectatus Linearitus



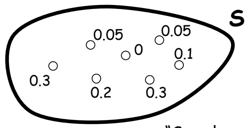
Random Variable

To use this new tool, we will also need to understand the concept of a <u>Random Variable</u>

Today's lecture: not too much material, but need to understand it well.

Probability Distribution

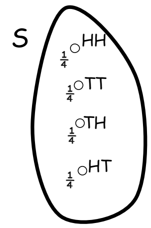
A (finite) probability distribution D
- a finite set S of elements (samples)
- each x∈S has weight or probability p(x) ∈ [0,1]



weights must sum to 1

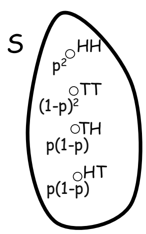
"Sample space"

Flip penny and nickel (unbiased)

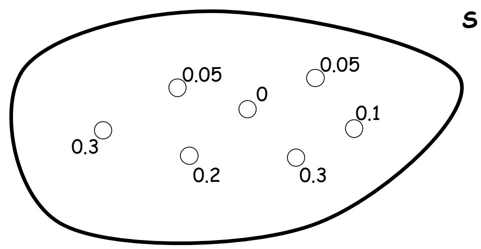


Flip penny and nickel (biased)

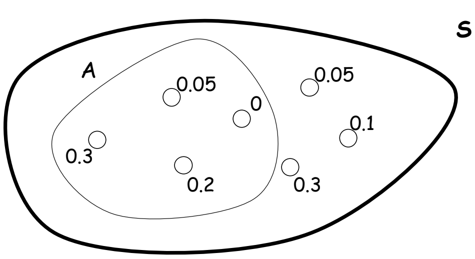
heads probability = p



Probability Distribution



An event is a subset



$Pr[A] = \sum_{x \in A} p(x) = 0.55$

Running Example

I throw a white die and a black die.

Sample space S =
{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
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Pr(x) = 1/36
∀ x ∈ S

E = event that sum ≤ 3

Pr[E] = |E|/|S| = proportion of E in S = 3/36

Axioms:

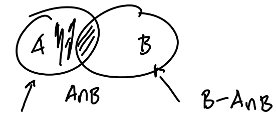
1) $P(A) \geq 0$ (positivity)

2) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$ (additivity)

$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ $\quad A_i \cap A_j = \emptyset$

3) $P(S) = 1$ (normalization)

$P(A \cup B)$



by additivity

$A - A\cap B = A \cap (A \cap B)^c$

$P(A \cup B) = P(A - A\cap B) + P(A \cap B) + P(B - A\cap B)$

$= P(A) - P(A\cap B) + P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \leq P(A) + P(B)$ "union bound"

New concept: <u>Random Variables</u>

Random Variables

Random Variable: a (real-valued) function on S

Examples:

X = value of white die.
  X(3,4) = 3,   X(1,6) = 1 etc.

Y = sum of values of the two dice.
  Y(3,4) = 7,   Y(1,6) = 7 etc.

W = (value of white die)$^{\text{value of black die}}$
  W(3,4) = 3^4 Y(1,6) = 1^6

Z = (1 if two dice are equal, 0 otherwise)
  Z(4,4) = 1,   Z(1,6) = 0 etc.

Toss a white die and a black die.

Sample space S =
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E.g., tossing a fair coin n times

S = all sequences of $\{H, T\}^n$

D = uniform distribution on S
  $\Rightarrow D(x) = (\frac{1}{2})^n$ for all $x \in S$

Random Variables (say n = 10)

X = # of heads
  X(HHHTTHTHTT) = 5

Y = (1 if #heads = #tails, 0 otherwise)
  Y(HHHTTHTHTT) = 1, Y(THHHHTTTTT) = 0

Notational conventions

Use letters like A, B, E for events.

Use letters like X, Y, f, g for R.V.'s.

R.V. = random variable

Two views of random variables

Think of a R.V. as

input to the function is random

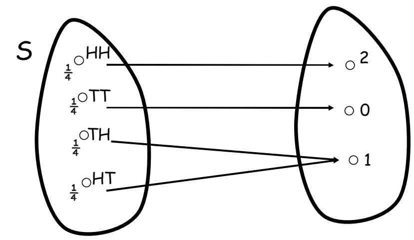
• a function from S to the reals $\mathbb{R}$

• or think of the induced distribution on $\mathbb{R}$

randomness is "pushed" to the values of the function

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads



Two views of random variables

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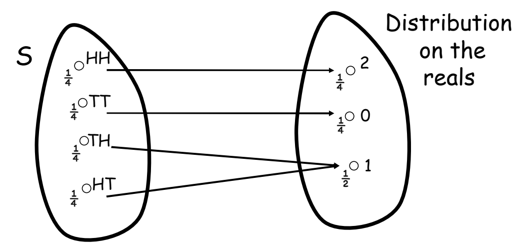

Two views of random variables

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Two dice

I throw a white die and a black die.

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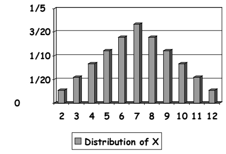
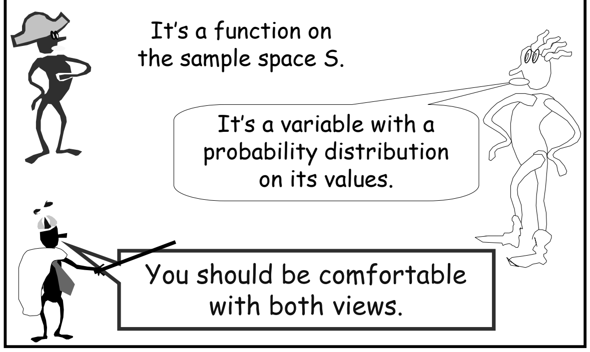


X = sum of both dice

function with X(1,1) = 2, X(1,2)=X(2,1)=3, …, X(6,6)=12

It's a floor wax *and* a dessert topping



From Random Variables to Events

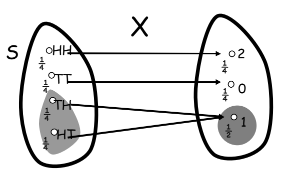
For any random variable X and value a,
we can define the <u>event</u> A that X=a.

Pr(A) = Pr(X=a) = Pr({x ∈ S| X(x)=a}).

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads

Pr(X = a) =
Pr({x ∈ S| X(x) = a})



Distribution
on X

Pr(X = 1)

= Pr({x ∈ S| X(x) = 1})

= Pr({TH, HT}) = ½.

Two dice

I throw a white die and a black die. X = sum

Sample space S =
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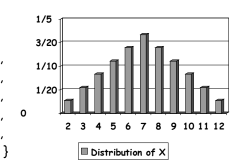


Pr(X = 6)
= Pr({x ∈ S | X(x) = 6})
= 5/36.

Pr(X = a) =
Pr({x ∈ S| X(x) = a})

From Random Variables to Events

For any random variable X and value a,
we can define the <u>event</u> A that X=a.

Pr(A) = Pr(X=a) = Pr({x ∈ S| X(x)=a}).

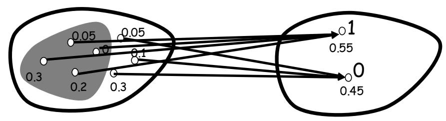
X has a
distribution on
its values

X is a function
on the sample space S

From Events to Random Variables

For any event A, can define the <u>indicator random variable</u> for A:

$X_A(x)$ = 1 if x ∈ A
= 0 if x ∉ A



Definition: expectation

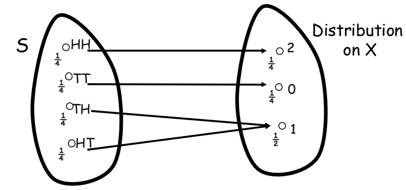
The <u>expectation</u>, or <u>expected value</u> of a random variable X is written as E[X], and is

$$E[X] = \sum_{x \in S} \Pr(x) X(x) = \sum_k k \Pr(X = k)$$

X is a function on the sample space S

X has a distribution on its values

Thinking about expectation



$\sum_{x \in S} X(x) \Pr(x)$

$\sum_k k \Pr(X = k)$

E[X] = ¼*0 + ¼*1 + ¼*1 + ¼*2 = 1.

E[X] = ¼*0 + ½*1 + ¼*2 = 1.

A quick calculation…

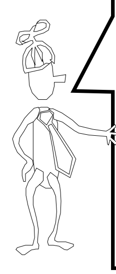
What if I flip a coin 3 times? Now what is the expected number of heads?

E[X] = (1/8) × 0 + (3/8) × 1 + (3/8) × 2 + (1/8) × 3 = 1.5

But Pr[X = 1.5] = 0…

Moral: don't always expect the expected.
Pr[X = E[X]] may be 0 !

Type checking



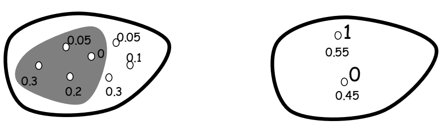
A Random Variable is the type of thing you might want to know an expected value of.

If you are computing an expectation, the thing whose expectation you are computing is a random variable.

Indicator R.V.s: $E[X_A] = \Pr(A)$

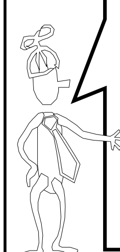
For event A, the <u>indicator random variable</u> for A:

$X_A(x)$ = 1 if $x \in A$
= 0 if $x \notin A$



$E[X_A] = 1 \times \Pr(X_A = 1) = \Pr(A)$

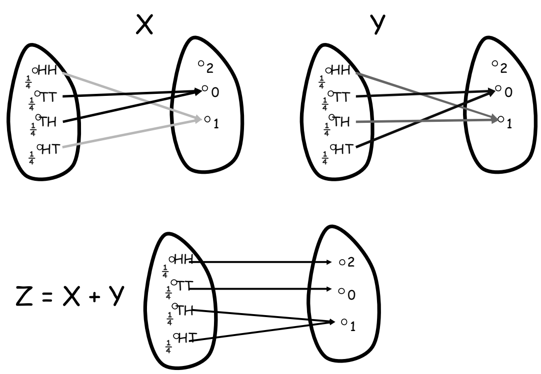
Adding Random Variables



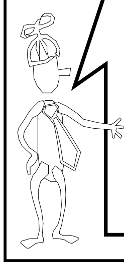
If X and Y are random variables (on the same set S), then Z = X + Y is also a random variable.

$$Z(x) \equiv X(x) + Y(x)$$

E.g., rolling two dice.
X = 1st die, Y= 2nd die,
Z = sum of two dice.



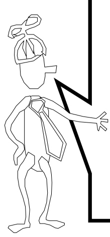
Adding Random Variables



Example: Consider picking a random person in the world. Let X = length of the person's left arm in inches. Y = length of the person's right arm in inches. Let Z = X+Y. Z measures the combined arm lengths.

Formally, S = {people in world}, D = uniform distribution on S.

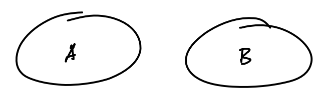
Independence



Two random variables X and Y are <u>independent</u> if for every a,b, the events X=a and Y=b are independent.

How about the case of X=1st die, Y=2nd die? X = left arm, Y=right arm?

$A \cap B = \phi$

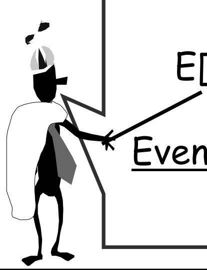


independent

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

Linearity of Expectation



If Z = X+Y, then

E[Z] = E[X] + E[Y]

<u>Even if X and Y are not independent</u>.

$$E(Z) = \sum_{x \in S} Pr(x)\, Z(x)$$

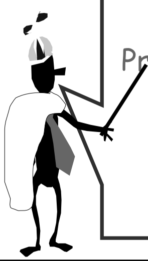
$$= \sum_{x \in S} Pr(x)\left(X(x) + Y(x)\right)$$

$$= \sum_{x \in S} Pr(x)\, X(x) + \sum_{x \in S} Pr(x)\, Y(x)$$

$$= E(X) + E(Y)$$

$$E(aX) = aE(X)$$

Linearity of Expectation



If Z = X+Y, then

$E[Z] = E[X] + E[Y]$

Proof:

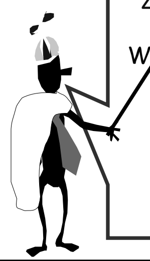
$E[X] = \sum_{x\in S} Pr(x)\, X(x)$

$E[Y] = \sum_{x\in S} Pr(x)\, Y(x)$

$E[Z] = \sum_{x\in S} Pr(x)\, Z(x)$

but $Z(x) = X(x) + Y(x)$

Linearity of Expectation

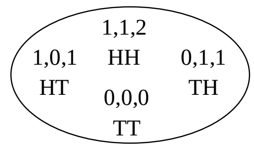


E.g., 2 fair flips:

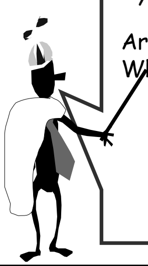
X = 1st coin, Y = 2nd coin.

Z = X+Y = total # heads.

What is E[X] ? E[Y] ? E[Z] ?



Linearity of Expectation



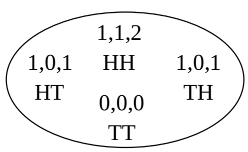
E.g., 2 fair flips:

X = at least one coin heads,

Y = both coins are heads, Z = X+Y

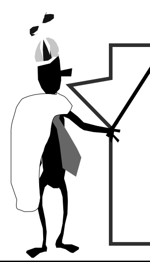
Are X and Y independent?

What is E[X] ? E[Y] ? E[Z] ?

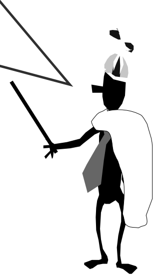


By induction

$E[X_1 + X_2 + \dots + X_n] =$
$E[X_1] + E[X_2] + \dots + E[X_n]$

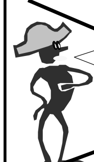


the expectation of the sum = the sum of the expectations



It is finally time to show off our probability prowess…

If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

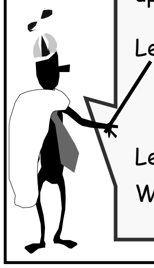


Hmm…

$\sum_k k\cdot Pr($*exactly* k letters end up in correct envelopes)

$= \sum_k k\cdot$ (…aarghll…)

Use Linearity of Expectation



Let A_i be the event the i^{th} letter ends up in its correct envelope.

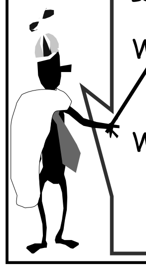
Let X_i be the indicator R.V. for A_i.

$$X_i = \begin{cases} 1 \text{ if } A_i \text{ occurs} \\ 0 \text{ otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$.

We are asking for E[Z].

Use Linearity of Expectation



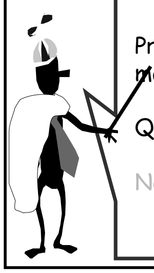
Let A_i be the event the i^{th} letter ends up in the correct envelope.
Let X_i be the indicator R.V. for A_i.
Let $Z = X_1 + \ldots + X_n$. We are asking for E[Z].

What is $E[X_i]$?
$E[X_i] = Pr(A_i) = 1/100$.

What is E[Z]?

$$E[Z] = E[X_1 + \ldots + X_{100}]$$
$$= E[X_1] + \ldots + E[X_{100}]$$
$$= 1/100 + \ldots + 1/100 = 1.$$

Use Linearity of Expectation



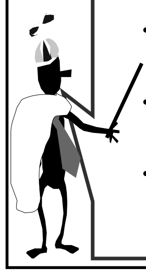
So, in expectation, 1 card will be in the same position as it started.

Pretty neat: it doesn't depend on how many cards!

Question: were the X_i independent?

No! E.g., think of n=2.

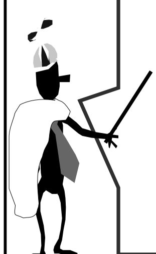
Use Linearity of Expectation



General approach:

- View thing you care about as expected value of some RV.
- Write this RV as sum of simpler RVs (typically indicator RVs).
- Solve for their expectations and add them up!

Example



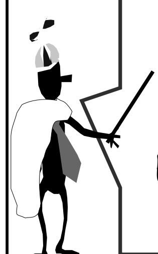
We flip n coins of bias p. What is the expected number of heads?

We could do this by summing

$\sum_k k \Pr(X = k)$

$= \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$

But now we know a better way



Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

$$X_i = \begin{cases} 0, \text{ if the } i^{th} \text{ coin is tails} \\ 1, \text{ if the } i^{th} \text{ coin is heads} \end{cases}$$

$$E[\,X\,] = E[\,\textstyle\sum_i X_i\,] = ?$$

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$E[\ X\ ] = E[\ \Sigma_i X_i\ ] = np$

What about Products?

If Z = XY, then
E[Z] = E[X] × E[Y]?

No!

X=indicator for "1st flip is heads"
Y=indicator for "1st flip is tails".

E[XY]=0.

But it is true if RVs are independent

$$E(Z) = \sum_c c\, P(Z=c)$$
$$= \sum_{a,b} a.b\, P(XY=c)$$
$$= \sum_{a,b} ab\, P(X=a \cap Y=b)$$
$$= \sum_{a,b} ab\, P(X=a)\, P(Y=b)$$
$$= \left(\sum_a a\, P(X=a)\right)\left(\sum_b b\, P(Y=b)\right)$$

But it is true if RVs are independent

Proof:

$E[X] = \Sigma_a\, a \times Pr(X=a)$
$E[Y] = \Sigma_b\, b \times Pr(Y=b)$

$E[XY] = \Sigma_c\, c \times Pr(XY = c)$
$= \Sigma_c \Sigma_{a,b:ab=c}\, c \times Pr(X=a \cap Y=b)$
$= \Sigma_{a,b}\, ab \times Pr(X=a \cap Y=b)$
$= \Sigma_{a,b} ab \times Pr(X=a)\, Pr(Y=b)$
$= E[X]\, E[Y]$

E.g., 2 fair flips.
X = indicator for 1st coin being heads,
Y = indicator for 2nd coin being heads.
XY = indicator for "both are heads".

$E[X] = \frac{1}{2}$, $E[Y] = \frac{1}{2}$, $E[XY] = \frac{1}{4}$.

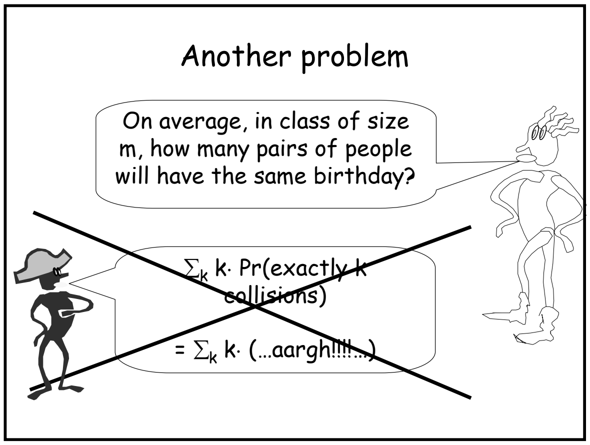
$E[X*X] = E[X]^2$?

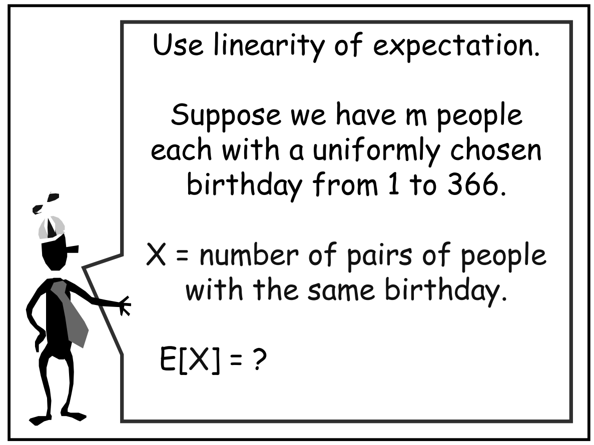
No: $E[X^2] = \frac{1}{2}$, $E[X]^2 = \frac{1}{4}$.

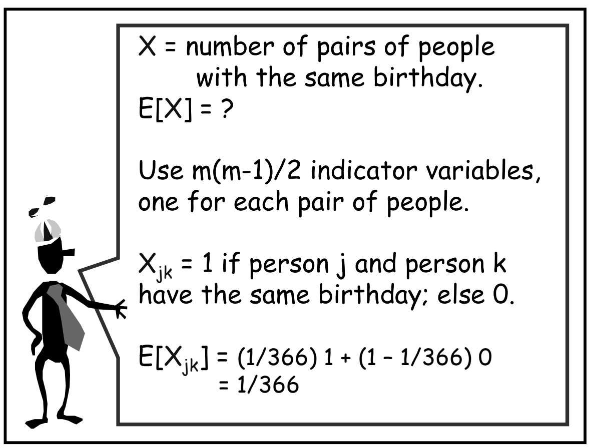
In fact, $E[X^2] - E[X]^2$ is called the *variance* of X.

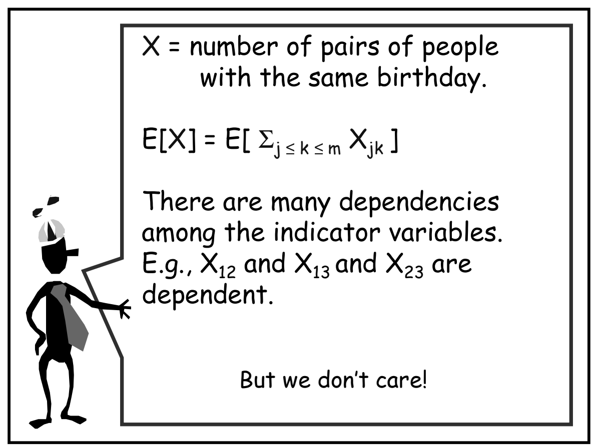
Most of the time, though, power will come from using sums.

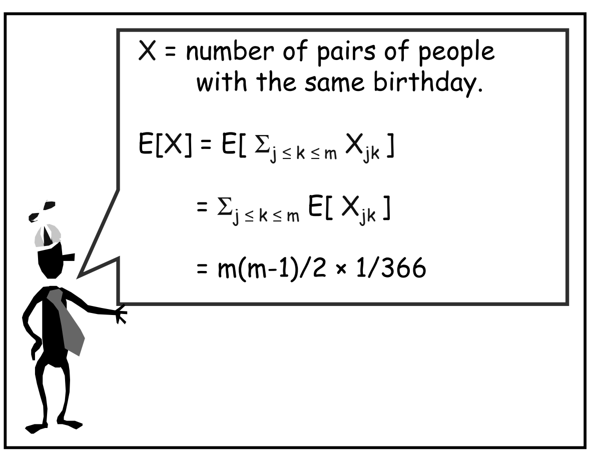
Mostly because Linearity of Expectations holds even if RVs are <u>not independent</u>.

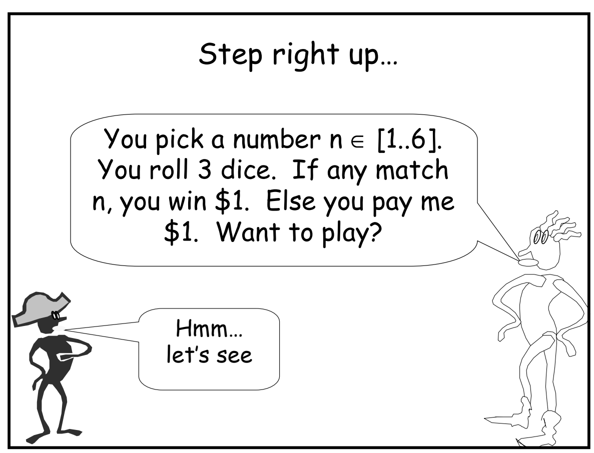

Another problem
On average, in class of size m, how many pairs of people will have the same birthday?
Σ_k k· Pr(exactly k collisions)
= Σ_k k· (…aargh!!!!…)


Use linearity of expectation.
Suppose we have m people each with a uniformly chosen birthday from 1 to 366.
X = number of pairs of people with the same birthday.
E[X] = ?


X = number of pairs of people with the same birthday.
E[X] = ?
Use m(m-1)/2 indicator variables, one for each pair of people.
X_{jk} = 1 if person j and person k have the same birthday; else 0.
$E[X_{jk}]$ = (1/366) 1 + (1 – 1/366) 0
= 1/366


X = number of pairs of people with the same birthday.
$E[X] = E[\ \Sigma_{j \le k \le m} X_{jk}\ ]$
There are many dependencies among the indicator variables.
E.g., X_{12} and X_{13} and X_{23} are dependent.
But we don't care!


X = number of pairs of people with the same birthday.
$E[X] = E[\ \Sigma_{j \le k \le m} X_{jk}\ ]$
$= \Sigma_{j \le k \le m} E[\ X_{jk}\ ]$
= m(m-1)/2 × 1/366


Step right up…
You pick a number n ∈ [1..6]. You roll 3 dice. If any match n, you win $1. Else you pay me $1. Want to play?
Hmm… let's see

Analysis

A_i = event that i-th die matches.

X_i = indicator RV for A_i.

Expected number of dice that match:

$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = \frac{1}{2}$.

But this is not the same as Pr(at least one die matches).

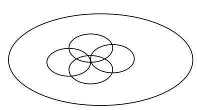
Analysis

Pr(at least one die matches)
= 1 – Pr(none match)
= $1 - (5/6)^3 = 0.416$.

What's going on?

Say we have a collection of events A_1, A_2, ….

How does E[# events that occur] compare to Pr(at least one occurs)?



What's going on?

E[# events that occur]
$= \sum_k \Pr(k \text{ events occur}) \times k$
$= \sum_{(k > 0)} \Pr(k \text{ events occur}) \times k$

Pr(at least one event occurs)
$= \sum_{(k > 0)} \Pr(k \text{ events occur})$

What's going on?

Moral #1: be careful you are modeling problem correctly.

Moral #2: watch out for carnival games.