CSE 321: Discrete Structures
Assignment #7
Due Friday, Dec 2

Reading Assignment: Read Sections 5.3, 7.1, 7.4, 7.5, and chapter 8.

Problems: You do **not** need to simplify any of your answers.

1. (6 points) A deck of 10 cards, each bearing a distinct number from 1 to 10, is shuffled to mix the cards thoroughly, so that each order is equally likely. What is the probability that the top three cards are in sorted (increasing) order?
2. (5 points) Suppose that A and B are events in a probability space, and that $Pr(A) = 0.5$, $Pr(B) = 0.2$ and $Pr(A \cup B) = 0.6$. What is $Pr(A \cap B)$?
3. (8 points) A biased coin with probability $3/5$ of coming up heads is flipped independently n times. What is the probability that all the heads occur at the end of the sequence?
4. (6 points) What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?
5. (6 points) Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?
6. (6 points) Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has
   (a) exactly three boys?
   (b) at least one boy?
   (c) all children of the same sex?
7. (8 points) Suppose two fair 6-sided dice are rolled independently. Let Y be the random variable which is the square of the sum of the two values showing. (So for example, if one die shows a 3 and the other shows a 4, then the value of the random variable is 49.) What is the expectation of Y? Let Z be the random variable which is the maximum of the two values showing. What is the expectation of Z? You do not need to simplify any of your answers (i.e., leave them in the form of a sum).
8. (8 points) Suppose that a fair coin is tossed 100 times. Let X be the random variable which is the number of flips i in which the coin takes on the same value in both flip i and $i + 1$. What is the expected value of X? (So for example in the sequence HHHH, X is 3, because the coin takes on the same value in positions 1 and 2, 2 and 3, and 3 and 4. In the sequence

THHHTT, the X is also 3 because the coin takes on the same value in positions 2 and 3, 3 and 4, and 5 and 6.) Hint: use indicator variables and linearity of expectation.

9. (8 points) Determine whether the following relations R on the set of all people are reflexive, symmetric, antisymmetric and/or transitive where $(a, b) \in R$ if and only if
   - a is taller than b
   - a and b were born on the same day
   - a has the same first name as b
   - a and b have a common grandparent.

10. (6 points) Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

11. (6 points) A relation R is called *circular* if $a\ R\ b$ and $b\ R\ c$ imply that $c\ R\ a$. Show that R is reflexive and circular if and only if it is an equivalence relation.

12. **Extra Credit** (10 points) 100 people walk into an airplane, each with a preassigned seat. The first person to walk in, rather than sitting in their own assigned seat, chooses one of the 100 seats uniformly at random and sits there. The remaining passengers walk onto the airplane one by one, and each person who walks in goes to their assigned seat. If nobody is sitting there, the passenger sits down. Otherwise, the passenger picks an empty seat uniformly at random and sits there. What is the probability that the last person will sit in his/her assigned seat? What is the expected number of people that end up in their assigned seat?

13. **Extra Credit** (8 points) You are driving through rural Washington, so bored out of your mind that you are reduced to counting out-of-state license plates. For this problem, let us assume that an out-of-state car is equally likely to be from any of the 49 other states. If you have just seen 10 out-of-state cars, what is the expected number of distinct (non-Washington) states you've seen license plates from?

14. **Extra Credit** (6 points) Let R be a random relation on a set $A = \{a_1, a_2, \ldots, a_n\}$ selected as follows: Independently, for each pair i, j, $1 \leq i \leq n$ and $1 \leq j \leq n$, (a_i, a_j) is included in R with probability $p_i \cdot p_j$. What is the probability that R is reflexive? What is the probability that R is symmetric?