CSE 321: Discrete Structures Spring 2004

PROBLEM SET 7

Due Friday, May 28, 2004, in class

All exercise numbers refer to the number in Rosen's book, 5th Edition.

1. Suppose that n balls are tossed into b bins so that each ball is equally likely to fall into any of the bins and that the tosses are independent.
   (a) Find the probability that a particular ball lands in a specified bin.
   (b) What is the expected number of balls that land in a particular bin?
   (c) What is the expected number of balls tossed until a particular bin contains a ball?
   (d) What is the expected number of balls tossed until all bins contain a ball?

2. Let E, F be events with $P(F) \neq 0$. Prove that

$$P(E) = P(E|F)P(F) + P(E|\bar{F})P(\bar{F}) \ .$$

3. Section 7.1, Exercise 4.

4. For the relation $R = \{(b,c), (b,e), (c,e), (d,a), (e,b), (e,c)\}$ on $\{a,b,c,d,e\}$, draw the following relations in digraph form:
   (a) The reflexive closure of R.
   (b) The symmetric closure of R.
   (c) The transitive closure of R.
   (d) The reflexive, symmetric, transitive closure of R.

5. Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation. (Can you identify what familiar objects the equivalence classes correspond to?)

6. A relation R is called *circular* if $a\ R\ b$ and $b\ R\ c$ imply that $c\ R\ a$. Show that R is reflexive and circular if and only if it is an equivalence relation.

7. Let R be a random relation on the set $A = \{a_1, a_2, \ldots, a_n\}$ selected as follows: Independently for each pair i, j, $1 \leq i \leq n$ and $1 \leq j \leq n$, include (a_i, a_j) in R with probability $1/2$. Now,
   (a) What is the probability that R is reflexive?
   (b) What is the probability that R is irreflexive? (A relation R on A is said to be *irreflexive* if for every $a \in A$, $(a,a) \notin R$.)
   (c) What is the probability that R is symmetric?
   (d) What is the probability that R is anti-symmetric?
   (e) * **(Bonus)** What is the probability that R is a *transitive tournament*, that is R is irreflexive, transitive, and for each $i \neq j$, exactly one of the pairs (a_i, a_j) and (a_j, a_i) is in R?