CSE 321
Homework 3

1

Let's assume that the statement is false. This means that for every pair of the 10 chosen numbers, the largest one is greater than twice the smaller one. Let's order the 10 chosen numbers, smallest to largest, and call them $x_1, x_2, \ldots, x_{10}$.

It's trivial that $x_1 \geq 1$. Since (x_1, x_2) is a pair, x_2 is greater than $2x_1$. Using $x_1 \geq 1$, we get that $2x_1 \geq 2$, therefore $x_2 > 2x_1 \geq 2$ or $x_2 > 2$ which means that $x_2 \geq 3$.

Using the same argument we can see that $x_3 > 2x_2 \geq 6, x_3 \geq 7$. Continuing in the same way

$$\begin{aligned}
x_4 > 2x_3 \geq 14 \quad &, \quad x_4 \geq 15 \\
x_5 > 2x_4 \geq 30 \quad &, \quad x_5 \geq 31 \\
x_6 > 2x_5 \geq 62 \quad &, \quad x_6 \geq 63 \\
x_7 > 2x_6 \geq 126 \quad &, \quad x_7 \geq 127 \\
x_8 > 2x_7 \geq 254 \quad &, \quad x_8 \geq 255 \\
x_9 > 2x_8 \geq 510 \quad &, \quad x_9 \geq 511 \\
x_{10} > 2x_9 \geq 1022 \quad &, \quad x_{10} \geq 1023
\end{aligned}$$

We reach the conclusion that $x_{10} \geq 1023$ which is a contradiction since the numbers are supposed to be less than or equal to 1000. Therefore the statement is true. $\square$

2

By induction

$$P(n) \longleftrightarrow \neg(p_1 \vee p_2 \vee \cdots \vee p_n) \Leftrightarrow \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$$

- For $n = 1$, $P(1)$ holds since $\neg(p_1) \Leftrightarrow \neg p_1$

- Assume $P(k)$ holds, or

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_k) \Leftrightarrow \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k$$

- To prove $P(k+1)$

$$\begin{aligned}
\neg(p_1 \vee p_2 \vee \cdots \vee p_k \vee p_{k+1}) &\Leftrightarrow \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \bigvee p_{k+1}) \\
&\Leftrightarrow \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \bigwedge \neg p_{k+1} &&\text{, De Morgan's Law} \\
&\Leftrightarrow \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \bigwedge \neg p_{k+1} &&\text{, Induction hypothesis}
\end{aligned}$$

$P(k+1)$ holds.

□

3

$$x^2 - 7y^2 = 3 \Leftrightarrow x^2 = 7y^2 + 3 \Rightarrow x^2 \equiv 3 \ mod 7$$

Therefore all solutions for x must satisfy $x^2 \equiv 3 \ mod7$. Any integer can be congruent to 0,1,2,3,4,5,6 modulo 7. Let's check each case

$x \ mod7$	$x^2 \ mod7$
0	0
1	1
2	4
3	2
4	2
5	4
6	1

We see that there is no case where $x^2 \equiv 3 \ mod7$ therefore no integer solution exists. □

4

By induction on $P(n) = \frac{n^2+n+2}{2}$

- For the case $n = 1$, there is just one line which clearly creates 2 regions.

- Assume that for k lines, the number of regions is $P(k) = \frac{k^2+k+2}{2}$

- Let's consider the case of $k+1$ lines. Pick any line x. That line must intersect every other line in different points in the plane, since no lines are parallel or interest at the same point.

  Consider what happens between two consecutive intersection points. If you don't consider line x there is a region there created by the other k lines. Now, line x splits that region into 2 regions. Therefore between 2 consecutive intersections line x adds an extra region to the total. There are $n-1$ pairs of consecutive intersection points. Furthermore, line x creates 2 extra regions, one before the first intersection and one after the last one. Therefore the total number of extra regions is $k+1$.

The total number of regions for $k+1$ lines is

$$\begin{aligned} P(k)+(k+1) &= \frac{k^2+k+2}{2}+k+1= \\ &= \frac{k^2+k+2+2k+2}{2}= \\ &= \frac{(k^2+2k+1)+k+1+2}{2}= \\ &= \frac{(k+1)^2+(k+1)+2}{2}= \\ &= P(k+1) \end{aligned}$$

□

5

In order to prove the statement we are going to prove that

$$P(n) = 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$$

By induction.

- $P(1) = 1 \leq 1 = 2-\frac{1}{1}$
- Assume that $P(k) \leq 2-\frac{1}{k}$

- $$\begin{aligned} P(k+1) &= 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} \\ &= P(k)+\frac{1}{(k+1)^2} \\ &\leq 2-\frac{1}{k}+\frac{1}{(k+1)^2} \qquad \text{, Induction hypothesis} \\ &= 2-\frac{(k+1)^2-k}{k(k+1)^2} \\ &= 2-\frac{k^2+2k+1-k}{k(k+1)^2} \\ &= 2-\frac{k^2+k+1}{k(k+1)^2} \\ &= 2-\frac{k(k+1)+1}{k(k+1)^2} \\ &= 2-\frac{k(k+1)}{k(k+1)^2}+\frac{1}{k(k+1)^2} \\ &= 2-\frac{1}{k+1}+\frac{1}{k(k+1)^2} \\ &\leq 2-\frac{1}{k+1} \qquad , \frac{1}{k(k+1)^2} \geq 0 \end{aligned}$$

This concludes the proof for $P(n) \leq 2-\frac{1}{n}$. But since $2-\frac{1}{n} < 2$, we conclude that $P(n) < 2$ □

6

Consider the prime factorization of 100!. For every 0 at the end of 100! there must be both a 2 and a 5 in the prime factorization. It is also the case that for every pair of 2 and 5 in the prime factorization we're going to get a 0 at the end. Clearly, the number of 0s is equal to the smallest power of 2 or 5 in the prime factorization.

2 is contained in all even numbers (we got 50 of them). 5 is contained once for every multiple of 5, namely $5, 10, 15, \ldots, 100$. It is contained twice in $25, 50, 75, 100$. The total number of 5s is therefore 24 and that is the number of 0s at the end.

7

Since $a \equiv b (mod\ m)$, $a = km + b$ for some k. Consider d, any common divisor of b and m. Clearly $d|km$ and $d|b$, therefore $d|a$. Similarly if we write $b = a - km$ and consider d' any common divisor of a and m, we can see that $d'|b$.

Therefore divisors of a, m are common with divisors of b, m and so must be the greatest one, therefore $gcd(a, m) = gcd(b, m)$. □