Supplementary Logic Notes
CSE 321 Spring 2001
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1 Propositional Logic

1.1 More efficient truth table methods

The method of using truth tables to prove facts about propositional formulas can be a very tedious procedure, especially if the formulas contain many propositional variables. Often, work can be saved because partial information about the truth values of subformulas of a formula can make work on the other parts of the formula unnecessary. For example, the formula $(G \vee H)$ will be a tautology if G is a tautology, no matter what the truth value of H is. If it is recognized that this is the case then much work can be saved, especially if G contains fewer variables that H does. An example using similar facts about $\rightarrow$ gives:

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$((p \vee q) \vee \neg r) \rightarrow (p \rightarrow (p \vee q))$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	T	T

and we see that this formula is a tautology since $(G \rightarrow H)$ is always true if H is true.

Even with these savings over a full truth table, in general the size of the table gets large very quickly as the number of propositional variables increases. It is also hard to determine in general whether or not one of these shortcuts will actually be successful. Despite this, many of the methods for proving propositional statements by computer are still directed at looking at all of the possible ways that a formula may be made true. Their format is designed to try and avoid unnecessary expansion of the full truth table of the formula but there frequently are bad examples for these methods too.

1.2 Natural Deduction

One of the reasons for looking at formal logic is to try to formalize the processes by which people reason about the world. One formal method for dealing with logical expressions that seems to match the kind of reasoning that people often use is called a *natural deduction system*. A natural deduction system consists of two parts, *equivalences* which are ways of re-expressing the same statement in another form, and *inference rules* which allow the derivation of statements that logically follow from a given statement or group of statements but are not necessarily equivalent expressions. We will represent our derivations in a formal way with each formula written on a different line and the reasons for deriving it from previous statements (either an equivalence or inference rule) described beside it. We will develop something akin to a programming language for giving proofs.

Equivalences	
Idempotent Laws	$p \lor p \iff p$ $p \land p \iff p$
Commutative Laws	$p \lor q \iff q \lor p$ $p \land q \iff q \land p$
Associative Laws	$(p \lor q) \lor r \iff p \lor (q \lor r)$ $(p \land q) \land r \iff p \land (q \land r)$
Distributive Laws	$p \lor (q \land r) \iff (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \iff (p \land q) \lor (p \land r)$
De Morgan's Laws	$\neg(p \land q) \iff \neg p \lor \neg q$ $\neg(p \lor q) \iff \neg p \land \neg q$
Double Negation Law	$\neg\neg p \iff p$
Contrapositive Law	$p \to q \iff \neg q \to \neg p$
Law of Implication	$p \to q \iff \neg p \lor q$

Table 1: Basic Propositional Equivalences

Given two logical formulas G and H, we will use the notation $G \Rightarrow H$ to denote the fact that assuming G we can logically derive H. Similarly we use $G \iff H$ to denote that fact that G and H are logically equivalent formulas. Note that $\Rightarrow$ and $\iff$ cannot be used in a logical formula but just describe relationships between two logical formulas. Table 1 gives a list of the basic propositional equivalences we will use.

For each propositional operator there is an inference rule that *adds* that operator and an inference rule that *eliminates* that operator. The inference rules are described in general in the following form

$$\frac{G_1,\ G_2,\ \ldots,\ G_k}{\therefore H_1,\ H_2,\ \ldots,\ H_l}$$

which is interpreted as meaning that if formulas matching all of G_1 through G_k occur on preceding lines then any of the formulas H_1 through H_l can be written out as a new line in the proof (where the same pattern matchings are used for H_1 through H_l as for G_1 through G_k – this may seem to be a mouthful but this will seem natural once we have seen some examples).

Table 2 lists the main propositional inferences we will use. The textbook contains a similar list but the lists of equivalences and inferences listed here are much more convenient for using in proofs than those in the text. (In fact it is not clear that the ones in the text are sufficient to prove all provable statements whereas the ones given here are sufficient, although we won't prove that they are.) The rules themselves (although not their names) are all fairly easy to remember except maybe the rule for eliminating $\vee$ – disjunctive syllogism. In fact, using the equivalences such as the Law of Implication, Modus Ponens can be used whenever it seems needed anyway. After the table we will discuss each kind of rule in detail and gives examples of their uses.

Inferences		
Modus Ponens	[e $\rightarrow$]	$\dfrac{p,\ p \rightarrow q}{\therefore q}$
Direct Proof	[a $\rightarrow$]	$\dfrac{p \Rightarrow q}{\therefore p \rightarrow q}$
Simplification	[e $\wedge$]	$\dfrac{p \wedge q}{\therefore p,\ q}$
Consolidation	[a $\wedge$]	$\dfrac{p,\ q}{\therefore p \wedge q}$
Disjunctive Syllogism	[e $\vee$]	$\dfrac{p \vee q,\ \neg p}{\therefore q}$
Addition	[a $\vee$]	$\dfrac{p}{\therefore p \vee q,\ q \vee p}$
Excluded Middle		$\dfrac{}{\therefore p \vee \neg p}$

Table 2: Basic Propositional Inferences
For each connective there is a rule for *adding* (a) or *eliminating* (a) that connective

Rules for $\rightarrow$:

Modus Ponens (Law of Detachment)

$$\frac{p,\ p \rightarrow q}{\therefore q}$$

This rule allows us to eliminate $\rightarrow$ from formulas and is one that dates back to the ancient Greeks. For example, suppose that we are given the fact that the following three formulas are true p, $p \rightarrow q$, and $q \rightarrow r$. We conclude that r must also be true as follows:

1. p	Given
2. $p \rightarrow q$	Given
3. $q \rightarrow r$	Given
4. q	Modus Ponens from 1 & 2
5. r	Modus Ponens from 4 & 3.

Notice the form of the derivation. We number each line and give the exact reason why each line followed from the previous ones. We can incorporate the equivalences too, as in the following example which shows that q follows from p and $(p \vee p) \rightarrow q$:

1. p	Given
2. $(p \vee p) \rightarrow q$	Given
3. $p \rightarrow q$	Idempotent Law from 2
4. q	Modus Ponens from 3

Notice that we can use the equivalences even on sub-formulas of a formula. However, when we use inference rules we must be dealing with the *entire* formula. If we didn't restrict our use of inference rules we could get into serious trouble and end up claiming things that weren't true.

Direct Proof

$$\frac{p \Rightarrow q}{\therefore p \rightarrow q}$$

This rule is used in a different way from the other ones we will state so we will postpone the explanation of exactly how it is used until after we discuss the remainder of the inference rules. Intuitively however its sense is clear: if we can derive q from p then $p \rightarrow q$ must be a true formula.

Rules for $\wedge$:

Consolidation

$$\frac{p,\ q}{\therefore p \wedge q}$$

Simplification

$$\frac{p \wedge q}{\therefore p,\ q}$$

These two rules for $\wedge$ are not hard to remember and are obviously correct. We will give one simple example which shows that r follows from $p \wedge q$ and $p \to r$:

1. $p \wedge q$	Given
2. $p \to r$	Given
3. p	Simplification from 1
4. r	Modus Ponens from 3 & 2.

Rules for $\vee$:

Addition

$$\frac{p}{\therefore p \vee q,\ q \vee p}$$

This follows from the simple fact that once we know that one part of an $\vee$ is true then the truth of the $\vee$ automatically follows. We didn't have to write both $p \vee q$ and $q \vee p$ since this is already handled by the commutative law for $\vee$ but doing so just makes things more convenient.

Disjunctive Syllogism

$$\frac{p \vee q,\ \neg p}{\therefore q}$$

This rule just describes the fact that if one of the disjuncts in an $\vee$ is false then the other must be true in order for the $\vee$ to be true.

Law of Excluded Middle:

$$\frac{}{\therefore p \vee \neg p}$$

This rule states a fundamental property of propositional logic, namely that every formula is either true or false. It is unique among our inference rules in that it does not depend on any previous statement. This is simply a fact that we can pull out of thin air whenever we need it.

Biconditionals:

The rules above are those needed to handle the four propositional operators, $\neg$, $\vee$, $\wedge$, and $\rightarrow$. If we also want to handle the biconditional operator $\leftrightarrow$ we add the following obvious inference rules:

Biconditional Consolidation

$$\frac{p \rightarrow q,\ q \rightarrow p}{\therefore p \leftrightarrow q}$$

Biconditional Simplification

$$\frac{p \leftrightarrow q}{\therefore p \rightarrow q,\ q \rightarrow p}$$

The Direct Proof Rule Revisited

As noted above, the direct proof rule is different from the others in that it refers to $\Rightarrow$, which says something about what we are able to prove rather than simply about what propositions we know. We use this rule as follows. In order to show $p \Rightarrow q$, we add a special line to our proof in which we assume p to be true. We then use all the laws of inference and all the propositions that we have on previous lines in order to come up with a line that says that q must be true. Thus we have shown that q follows from p and our the previous information, so $p \Rightarrow q$ holds and we can conclude, using the direct proof rule, that $p \rightarrow q$ follows. Notice that the entire portion of the proof between when we assumed p and when we derived q may depend on the assumption that p is true. Thus this portion of the proof is off limits to the rest of the proof.

You may notice that the way we want to use the direct proof rule is sort of like setting up a little *subroutine* or *block* that does the business of proving $p \rightarrow q$. In fact we have rules of scope just like nested blocks or subroutines: All propositions previously proved from outside the block are available inside the block but, once we have popped out of the block, all the information about what was going on inside disappears. To show that we are inside a block we will indent the list of propositions; the first line of this block will always be the assumption p and the last line will always be the conclusion q. Only the result, the implied proposition $p \rightarrow q$, is listed at the outer level immediately following this block. An example should make this clear; we show that $(p \wedge q) \rightarrow r$ follows from $p \rightarrow (q \rightarrow r)$.

1. $p \rightarrow (q \rightarrow r)$	Given
2. $p \wedge q$	Assumption
3. p	Simplification from 2
4. q	Simplification from 2
5. $q \rightarrow r$	Modus Ponens from 3 & 1
6. r	Modus Ponens from 4 & 5
7. $(p \wedge q) \rightarrow r$	Direct Proof Rule

In this example, when we popped out of the nested block we finished the proof. In general there may be many other deductions after a block is finished as we will see in several examples below.

1.3 Some Derived Inferences

The remainder of this section on propositional logic will be taken up with some inferences that can be derived from those previously stated. You will notice that a number of these are inference rules listed in the text. You will also notice how a number of them correspond to the methods of proof listed in the text. Where appropriate we will give them names and make use of them in future inferences but these names are not really important in themselves.

Hypothetical Syllogism

$$\frac{p \to q,\ q \to r}{\therefore p \to r}$$

1. $p \to q$	Given
2. $q \to r$	Given
3. p	Assumption
4. q	Modus Ponens from 3 & 1
5. r	Modus Ponens from 4 & 2
6. $p \to r$	Direct Proof Rule

Modus Tollens

$$\frac{\neg q,\ p \to q}{\therefore \neg p}$$

1. $\neg q$	Given
2. $p \to q$	Given
3. $\neg q \to \neg p$	Contrapositive from 2
4. $\neg p$	Modus Ponens from 1 & 3

Reductio ad Absurdum

$$\frac{p \to (q \wedge \neg q)}{\therefore \neg p}$$

1. $p \to (q \wedge \neg q)$	Given
2. $\neg(q \wedge \neg q) \to \neg p$	Contrapositive from 1
3. $\neg q \vee \neg\neg q$	Excluded Middle
4. $\neg(q \wedge \neg q)$	De Morgan's Law from 3
5. $\neg p$	Modus Ponens from 4 & 2

Trivial Proof

$$\frac{q}{\therefore p \to q}$$

1. q	Given
2. p	Assumption
3. q	Copy of line 1
4. $p \to q$	Direct Proof Rule

Vacuous Proof

$$\frac{\neg p}{\therefore p \rightarrow q}$$

1. $\neg p$ Given
2. $\neg p \vee q$ Addition from 1
3. $p \rightarrow q$ Equivalence from 2

Proof by Contradiction

$$\frac{\neg(p \wedge \neg q)}{\therefore p \rightarrow q}$$

1. $\neg(p \wedge \neg q)$ Given
2. $\neg p \vee \neg\neg q$ De Morgan's Law from 1
3. $\neg p \vee q$ Double Negation from 2
4. $p \rightarrow q$ Law of Implication from 3

Proof by Cases

$$\frac{p \vee q,\ p \rightarrow r,\ q \rightarrow r}{\therefore r}$$

1. $p \vee q$ Given
2. $p \rightarrow r$ Given
3. $q \rightarrow r$ Given
4. $\neg r \rightarrow \neg p$ Contrapositive from 2
5. $\neg r \rightarrow \neg q$ Contrapositive from 3
    6. $\neg r$ Assumption
    7. $\neg p$ Modus Ponens from 6 & 4
    8. $\neg q$ Modus Ponens from 6 & 3
    9. $\neg p \wedge \neg q$ Consolidation from 7 & 8
    10. $\neg(p \vee q)$ De Morgan's from 9
11. $\neg r \rightarrow \neg(p \vee q)$ Direct Proof Rule
12. $(p \vee q) \rightarrow r$ Contrapositive from 11
13. r Modus Ponens from 12

The following is the converse of the implication proven in the section on direct proof. It illustrates how doubly nested uses of the direct proof rule are handled.

$$\frac{(p \wedge q) \rightarrow r}{\therefore p \rightarrow (q \rightarrow r)}$$

1. $(p \wedge q) \rightarrow r$	Given
2. p	Assumption
3. q	Assumption
4. $p \wedge q$	Consolidation from 2 & 3
5. r	Modus Ponens from 4 & 1
6. $q \rightarrow r$	Direct Proof Rule
7. $p \rightarrow (q \rightarrow r)$	Direct Proof Rule

The following shows that any proposition follows from a contradiction.

$$\frac{p \wedge \neg p}{\therefore q}$$

1. $p \wedge \neg p$	Given
2. p	Simplification from 1
3. $\neg p$	Simplification from 1
4. $p \rightarrow q$	Vacuous Proof from 3
5. q	Modus Ponens from 2 & 4

2 Predicate Logic

2.1 Natural Deduction for Predicate Logic

Natural deduction for predicate logic is an extension of natural deduction for propositional logic. All the same propositional rules and equivalences apply. The only new equivalences are the laws of quantifier negation which, as we have seen, can be thought of as extensions of De Morgan's Laws for propositional logic.

Equivalences	
Quantifier Negation Laws	$\neg \exists x P(x) \iff \forall x \neg P(x)$
	$\neg \forall x P(x) \iff \exists x \neg P(x)$

Since there are two quantifiers, we have a total of four new predicate logic inference rules, one for adding and one for eliminating each of the quantifiers. In all of them we relate quantified expressions to propositions formed by substituting constants for the variables and removing the quantifiers. Constant symbols have the property of being either *arbitrary* or *specific*. Essentially a constant symbol of type arbitrary is appropriate for universally quantified expressions and a constant symbol of type specific is appropriate for existentially quantified expressions.

The type of a constant symbol is determined by the very first line in which it appears and is written in a special column of the proof on that line. When a constant symbol of type specific is first created, one also must list the other constant symbols that it might depend on. In general a specific constant symbol depends on all constants of type arbitrary that appear in in the line where it is defined, as well as all constants of type arbitrary that appear in any of the assumption lines of the direct proofs within which the statement is nested. Call this the dependent set of symbols. (We will see in examples how this is used.)

Certainly if $\forall x P(x)$ is true then, for any c we choose, $P(c)$ is true. Therefore we have:

Universal Instantiation

$$\frac{\forall x P(x)}{\therefore P(c) : \ c \text{ arbitrary (new or old)}}$$

If c has never been used before then the fact that c is a constant symbol of type arbitrary is written on the line where this rule is used. This corresponds to the typical form in proofs written in English which begin "Let c be any ..." and then proceed to reason about the properties of c. We can also use this rule for cases when c has already appeared previously in the proof in which case we don't mark the type of c.

Arbitrary constant symbols may also be introduced in the assumptions for direct proof rules.

Universal Generalization

$$\frac{P(c) : \ c \text{ arbitrary; no specific symbols depending on } c}{\therefore \forall x P(x)}$$

This follows since if $P(c)$ is true for an arbitrary c about which we have made no special conditions then certainly $\forall x P(x)$ is true. A common pattern in proofs of universal ($\forall$) statements is to begin with an assumption that involves some constant of type arbitrary, do some reasoning and then finish using universal generalization.

Existential Generalization

$$\frac{P(c) : \ c \text{ specific or arbitrary}}{\therefore \exists x P(x)}$$

If $P(c)$ is true for some c then $\exists x P(x)$ is true whether or not c is required to satisfy some special conditions.

Existential Instantiation

$$\frac{\exists x P(x)}{\therefore P(c) : \text{ } c \text{ new and specific; list of constants on which } c \text{ depends}}$$

If $\exists x P(x)$ is true then there is some value c that makes $P(c)$ true, but this value may have to satisfy some very special conditions. We don't necessarily know that it is the same as anything we've seen before. Thus we require that the constant symbol chosen when this rule is applied has not appeared in any previous line in the derivation. For example, if c has already been used then we use d and if that has been used we choose e, f, etc. This corresponds to the situation in proofs in English where we know $\exists x P(x)$ but we don't know which value of x makes $P(x)$ true and we want to talk about that value (whatever it is). This might appear as "Let c be the value of x that makes $P(x)$ true" or often we just say something like "Therefore $P(c)$ is true for some c".

Note that both instantiation rules apply to only one quantifier at a time. Even if the same variable is used with different quantifiers in the same formula, only one of the quantifiers is affected.

One rule of thumb for applying the instantiation (quantifier elimination) rules is to do the existential ones first. This is not essential but it is a more flexible way of doing things since every time we use existential instantiation we must choose a new constant symbol that we haven't used earlier in a proof.

The rules of instantiation and generalization are typically used in the following sort of context in proofs written in English:

Suppose one wanted to show that for every x, if x is an even number then $x+3$ is odd. This is a statement of the form $\forall x(E(x) \rightarrow O(x+3))$ where $E(x) = \exists y(x = 2y)$ means x is even and $O(x) = \exists y(x = 2y+1)$ means x is odd, in other words that $\forall x(\exists y(x = 2y) \rightarrow \exists y(x+1 = 2y+1))$. One might begin a direct proof of the fact in English as follows: Let n be any even number. Therefore $n = 2m$ for some m. Therefore $n + 3 = 2m + 3 = 2(m + 1) + 1$. It follows that $O(n + 3)$ is true as required.

$$\overline{\therefore \forall x(\exists y(x = 2y) \rightarrow \exists y(x + 3 = 2y + 1))}$$

1. $\exists y(n = 2y)$	n arbitrary	Assumption	
2. $n = 2m$		Exist. Inst. from 1	m specific; depends on n
3. $n + 3 = 2m + 3$		Math manip. from 2	
4. $n + 3 = 2(m + 1) + 1$		Math manip. from 3	
5. $\exists y(n + 3 = 2y + 1)$		Exist. Gen. from 4	
6. $\exists y(n = 2y) \rightarrow \exists y(n + 3 = 2y + 1)$		Direct Proof Rule	
7. $\forall x(\exists y(x = 2y) \rightarrow \exists y(x + 3 = 2y + 1))$		Univ. Gen. from 6	

Compare this with the proof written above in English. Line 1 begins with the assumption that n is an even number (it just spells out the definition). Line 2 emphasizes the fact that the value of m depends on the value of n. Lines 6 and 7 emphasize something that is implicit in the English proof, namely that we've used direct proof and are really generalizing using the fact that n was arbitrary.

This is an example of a proof method described in section 3.1 of the text as Direct Proof. This method is based on proving something of the form $p \rightarrow q$ by assuming p and deriving q (which our Direct Proof Rule is exactly set up to do).

2.2 Some Other Derived Inferences

$$\frac{\exists x \forall y P(x,y)}{\therefore \forall y \exists x P(x,y)}$$

1. $\exists x \forall y P(x,y)$	Given	
2. $\forall y P(c,y)$	Exist. Inst. from 1	c specific; no dep.
3. $P(c,d)$	Univ. Inst. from 2	d arbitrary
4. $\exists x P(x,d)$	Existential Gen. from 3	
5. $\forall y \exists x P(x,y)$	Universal Gen. from 4	

$$\frac{\forall x P(x) \wedge \forall x Q(x)}{\therefore \forall x (P(x) \wedge Q(x))}$$

1. $\forall x P(x) \wedge \forall x Q(x)$	Given	
2. $P(c) \wedge \forall x Q(x)$	Universal Inst. from 1	c arbitrary
3. $P(c) \wedge Q(c)$	Universal Inst. from 2	
4. $\forall x (P(x) \wedge Q(x))$	Universal Gen. from 3	

Here's an example of where the rules protect us from making an incorrect conclusion: Suppose we tried to show that $\exists x P(x) \wedge \exists x Q(x)$ implies that $\exists x (P(x) \wedge Q(x))$ which is actually an incorrect implication. (Can you come up with a situation which shows that it is incorrect?) We would begin our derivation as follows:

1. $\exists x P(x) \wedge \exists x Q(x)$	Given	
2. $P(c) \wedge \exists x Q(x)$	Exist. Inst. from 1	c specific; no dep.
3. $P(c) \wedge Q(d)$	Exist. Inst. from 2	d specific; no dep.

The Existential Instantiation rule prevents us from using the same c we did before when we get to line 3, unlike the situation for Universal Instantiation. The argument would get stuck at this point because there is no way to get c and d to be the same constant symbol which would be necessary to derive the desired conclusion.

Here is a non-trivial example that combines some of the propositional inference with predicate inference:

$$\frac{\exists x P(x) \rightarrow \exists x Q(x)}{\therefore \exists x (P(x) \rightarrow Q(x))}$$

1. $\exists x P(x) \rightarrow \exists x Q(x)$	Given	
2. $\neg \exists x P(x) \vee \exists x Q(x)$	Law of Implication from 1	
3. $\forall x \neg P(x) \vee \exists x Q(x)$	Quantifier Negation Law from 2	
4. $\forall x \neg P(x) \vee Q(c)$	Exist. Inst. from 3	c specific; no dep.
5. $\neg P(c) \vee Q(c)$	Universal Inst. from 4	
6. $P(c) \rightarrow Q(c)$	Law of Implication from 5	
7. $\exists x (P(x) \rightarrow Q(x))$	Existential Gen. from 6	

Note that although Universal Instantiation was used in line 5, c is a specific constant symbol because it was previously introduced in line 4. In this proof we used the Quantifier Negation Laws and the propositional rules to rewrite the formulas so that $\forall$ appears outermost where possible. This is a useful rule of thumb since there is more flexibility when dealing with constants with universal quantifiers.

Finally, here is an example where keeping track of what specific constants depend on is important. Suppose were trying to show that $\exists y \forall x P(x, y)$ follows from $\forall x \exists y P(x, y)$ which is not in general true. (Can you come up with a case when it isn't?) We would get the following:

1. $\forall x \exists y P(x, y)$	Given	
2. $\exists y P(c, y)$	Universal Inst. from 1	c arbitrary
3. $P(c, d)$	Existential Inst. from 2	d specific, depends on c

At this point in order to get $\exists y \forall x P(x, y)$ we would have to apply Universal Generalization to line 3. However we are blocked from doing so since c depends on d. Thus dependencies prevent incorrect inferences from being made.

2.3 A Note about Proofs

You should find the examples presented here to be fairly straightforward to check. In fact a proof is merely a sequence of assertions ending in what is to be proved, so that each assertion follows from previous ones using a rule from of a well-defined set of simple correct rules. Once these rules of reasoning have been set down it should be an easy process to see whether or not they have been followed. (In fact you could program a computer to check these proofs.) The process of actually coming up with proofs is a much harder task than that of checking them. This requires both practice and art.

Expressing statements in formal logic can help you to understand what precisely must be shown in order to solve a problem. My hope is that the kinds of reasoning that this system of natural deduction uses will help to guide you in figuring out what precisely must be done in order to solve problems and will help to clarify your thinking about them.

3 Appendix

Equivalences	
Idempotent Laws	$p \vee p \iff p$ $p \wedge p \iff p$
Commutative Laws	$p \vee q \iff q \vee p$ $p \wedge q \iff q \wedge p$
Associative Laws	$(p \vee q) \vee r \iff p \vee (q \vee r)$ $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$
Distributive Laws	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\neg(p \wedge q) \iff \neg p \vee \neg q$ $\neg(p \vee q) \iff \neg p \wedge \neg q$
Double Negation Law	$\neg\neg p \iff p$
Contrapositive Law	$p \to q \iff \neg q \to \neg p$
Implication Law	$p \to q \iff \neg p \vee q$
Quantifier Negation Laws	$\neg\exists x P(x) \iff \forall x \neg P(x)$ $\neg\forall x P(x) \iff \exists x \neg P(x)$

Propositional and Predicate Equivalences

Inferences		
Modus Ponens	[e $\to$]	$\dfrac{p,\ p \to q}{\therefore q}$
Direct Proof	[a $\to$]	$\dfrac{p \Rightarrow q}{\therefore p \to q}$
Simplification	[e $\wedge$]	$\dfrac{p \wedge q}{\therefore p,\ q}$
Consolidation	[a $\wedge$]	$\dfrac{p,\ q}{\therefore p \wedge q}$
Disjunctive Syllogism	[e $\vee$]	$\dfrac{p \vee q,\ \neg p}{\therefore q}$
Addition	[a $\vee$]	$\dfrac{p}{\therefore p \vee q,\ q \vee p}$
Excluded Middle		$\dfrac{}{\therefore p \vee \neg p}$
Universal Instantiation	[e $\forall$]	$\dfrac{\forall x P(x)}{\therefore P(c) :\ c \text{ arbitrary}}$
Universal Generalization	[a $\forall$]	$\dfrac{P(c) :\ c \text{ arbitrary; no dependency}}{\therefore \forall x P(x)}$
Existential Instantiation	[e $\exists$]	$\dfrac{\exists x P(x)}{\therefore P(c) :\ c \text{ new and specific; depends on ...}}$
Existential Generalization	[a $\exists$]	$\dfrac{P(c) :\ c \text{ specific or arbitrary}}{\therefore \exists x P(x)}$

Propositional and Predicate Inferences