Discrete Structures

Graphs

Chapter 7, Sections 7.1 - 7.3

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Undirected Graphs

◇ A **simple graph** $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges.

◇ A **multigraph** $G = (V, E)$ consists of a set V of vertices, a set E of edges, and a function f from E to $\{\{u, v\} \mid u, v \in V, u \neq v\}$. The edges e_1 and e_2 are called multiple or parallel edges if $f(e_1) = f(e_2)$.

◇ A **pseudograph** $G = (V, E)$ consists of a set V of vertices, a set E of edges, and a function f from E to $\{\{u, v\} \mid u, v \in V\}$. An edge is a loop if $f(e) = \{u, u\} = \{u\}$ for some $u \in V$.

Directed Graphs

- ◇ A **directed graph** $G = (V, E)$ consists of a set V of vertices and a set of edges E that are ordered pairs of elements of V.

- ◇ A **directed multigraph** $G = (V, E)$ consists of a set V of vertices, a set E of edges, and a function f from E to $\{(u, v) \mid u, v \in V\}$. The edges e_1 and e_2 are multiple edges if $f(e_1) = f(e_2)$.

Undirected Graph Terminology

- ◇ Two vertices u and v in an undirected graph G are called **adjacent** (or **neighbors**) in G if $\{u, v\}$ is an edge of G. If $e = \{u, v\}$, the edge e is called **incident with** the vertices u and v. The edge e is also said to **connect** u and v. The vertices u and v are called **endpoints** of the edges $\{u, v\}$.

- ◇ The **degree** of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

- ◇ **The Handshaking Theorem** : Let $G = (V, E)$ be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} \text{deg}(v).$$

- ◇ **Theorem** : An undirected graph has an even number of vertices of odd degree.

Directed Graph Terminology

◇ When (u, v) is an edge of the graph G with directed edges, u is said to be **adjacent to** v and v is said to be **adjacent from** u. The vertex u is called the **initial vertex** of (u, v), and v is called the **terminal** or **end vertex** of (u, v). The initial vertex and terminal vertex of a loop are the same.

◇ In a graph with directed edges the **in-degree** of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The **out-degree** of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex.

◇ **Theorem:** Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

More Definitions ...

- $\diamondsuit$ A simple graph is G is called **bipartite** if its vertex V can be partitioned into two disjoint nonempty sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2.

- $\diamondsuit$ A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.

- $\diamondsuit$ The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

- $\diamondsuit$ The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an **isomorphism**.

Connectivity 1

- A **path of lengh** n from u to v, where n is a positive integer, in an undirected graph is a sequence of edges $e_1, e_2, \ldots, e_n$ of the graph such that $f(e_1) = \{x_0, x_1\}, f(e_2) = \{x_1, x_2\}, \ldots f(e_n) = \{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$. The path is a **circuit** if it begins and ends at the same vertex. The path or circuit is said to pass through or traverse the vertices $x_1, x_2, \ldots, x_{n-1}$. A path or circuit is **simple** if it does not contain the same edge more than once.

- A **path of lengh** n from u to v in a directed multigraph, where n is a positive integer, is a sequence of edges $e_1, e_2, \ldots, e_n$ of the graph such that $f(e_1) = (x_0, x_1), f(e_2) = (x_1, x_2), \ldots f(e_n) = (x_{n-1}, x_n)$, where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the graph, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$. The path is a **circuit** or **cycle** if it begins and ends at the same vertex. A path or circuit is **simple** if it does not contain the same edge more than once.

Connectivity 2

- An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph.

- **Theorem:** There is a simple path between every pair of distinct vertices of a connected undirected graph.

- A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

- A directed graph is **weakly connected** if there is a path between any two vertices in the underlying undirected graph.

Euler Circuits

- An **Euler circuit** in a graph G is a simple circuit containing every edge of G.
- **Theorem:** A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.