Practice Questions
from previous CSE321 exams

1. If $D(x, y)$ is the predicate 'x divides y' then which of the following statements are true in the domain of positive integers?
   - $\forall x D(x, x)$.
   - $\forall x \forall y (D(x, y) \rightarrow D(y, x))$.
   - $\forall x \forall y ((D(x, y) \text{AND} D(y, x)) \rightarrow (x = y))$.
   - $\forall x \forall y (D(x, y) \text{OR} D(y, x))$
   - $\forall x \forall y \forall z ((D(x, y) \text{AND} D(y, z)) \rightarrow D(x, z))$.

2. Let $P(x, y)$ be the predicate "x is a parent of y", and let $O(x, y)$ be the predicate "x is older than y", and let the universe for all variables be the set of all people. Express each of the following statements as a predicate logic formula using P and O:
   - Every parent is older than his/her children.
   - Alice and Bob have the same parents.
   - John is Mary's oldest child.
   - Every person has at least two parents.
   - No two people are exactly the same age.

3. True or false:
   - $p \rightarrow q$ is logically equivalent to $q \rightarrow p$.
   - $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ is a tautology.
   - $((\forall x [P(x) \rightarrow Q(x)]) \wedge P(y)) \rightarrow Q(y)$ is a tautology.
   - There is a one-to-one function from A to B if and only if there exists an onto function from B to A.
   - To prove by contradiction that $p \rightarrow q$, one must show that p is false.
   - If A is a subset of B, then the power set of A is a subset of the power set of B.
   - For any two sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
   - The rationals are countable.
   - Every subset of a countable set is countable.

The next two true/false questions refer to functions: $g : A \to B$ and $f : B \to C$. (Remember the definition of the composition of two functions: $f \circ g(x) = f(g(x))$.) For each of the following, answer true if the statement is true for *all* f and g satisfying the conditions, and answer false otherwise.

- If f is one-to-one, $f \circ g$ is one-to-one.

- If f and g are both onto, then $f \circ g$ is onto.

4. Find $\gcd(2n + 1, 3n + 2)$, where n is a positive integer. Hint: Use the Euclidean algorithm.
5. Define a function g on the non-negative integers by $g(0) = 2$, $g(1) = 3$ and $g(n + 1) = 3g(n) - 2g(n-1)$ for all $n \geq 1$. Use strong induction to prove that for all $n \geq 0$, $g(n) = 2^n + 1$.
6. Prove by contradiction: If x is a rational number and y is an irrational number, then $x + y$ is irrational. You may use the following fact: if a and b are nonzero rational numbers, then a/b is also rational.
7. Prove by induction that

$$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots + n \cdot 2^{n-1} = (n - 1) \cdot 2^n + 1$$

whenever n is a positive integer.

8. Consider the following proof:

Claim: *Every natural number is either prime or a perfect square.*[1]

Proof: *We prove by induction that for all natural numbers n, $P(n)$: n is a prime or a perfect square.*
Base case: *$P(1)$ is certainly true.*
Inductive Hypothesis: *Every natural number less than n is a prime or a perfect square.*
Inductive step: *Consider n. If n is prime, then we are done. Otherwise, n can be factored as $n = rs$ with r and s less than or equal to $n - 1$. By the inductive hypothesis, r and s are perfect squares, so $r = u^2$ and $s = v^2$. Therefore, $n = rs = u^2v^2 = (uv)^2$. So n is a perfect square.*
Therefore every natural number is either prime or a perfect square.

- (4 points) Does this proof use regular induction or strong induction?

- (8 points) Which of the following statements are true? (There could be zero, one, or more than one true statement.)

[1] An integer a is a *perfect square* if there is an integer b such that $a = b^2$.

(a) The proof is wrong because the inductive hypothesis is applied incorrectly. The inductive hypothesis asserts that r and s are **either** perfect squares **or** primes, but the proof uses it to conclude that r and s are perfect squares, ignoring the possibility that they are primes.

(b) The proof is wrong because it proceeds by trying to prove that n is either a prime or a perfect square. But that is already the inductive hypothesis. Instead the proof should proceed by showing that $n + 1$ is either prime or a perfect square.

(c) The proof is wrong because it is incorrect to claim that "If n is prime, then we are done," because this is what we were trying to prove.

(d) The proof is correct.