Section 8: Tail Bounds

Review of Main Concepts

- **Markov's Inequality**: Let X be a non-negative random variable, and $\alpha > 0$. Then,

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

- **Chebyshev's Inequality**: Suppose Y is a random variable with $\mathbb{E}[Y] = \mu$ and $\text{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$\mathbb{P}(|Y - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$$

- **(Multiplicative) Chernoff Bound**: Let $X_1, X_2, ..., X_n$ be *independent* Bernoulli random variables.

  Let $X = \Sigma_{i=1}^n X_i$, and $\mu = \mathbb{E}[X]$. Then, for any $0 \leq \delta \leq 1$,

  - $\mathbb{P}(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$
  - $\mathbb{P}(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$

0. Central Limit Theorem Practice

You're playing ping pong with your friend, and want to keep playing until you've scored 15 points. Unfortunately, your friend is a much more skilled ping pong player than you, so you only win points 25% of the time (with each point being independent of the other points). Approximate the probability that you'll need to play at least 50 points before stopping.

1. Content Review

(a) True or false: the Union Bound always gives a result in $[0, 1]$.

(b) True or false: Markov's Inequality always gives a non-negative result.

(c) Suppose C and D are discrete random variables. Then $\mathbb{E}[C|D = d] =$

- ◯ $\sum_d d p_{D|C}(d|c)$
- ◯ $\sum_c c p_{C|D}(c|d)$
- ◯ $\int_{-\infty}^{\infty} c f_{c|d} dx$
- ◯ $\frac{\mathbb{E}[C]}{\mathbb{E}[D]}$

(d) Suppose X and Y are random variables and A is an event. Given that $\mathbb{E}[X|A] = 4$ and $\mathbb{E}[Y|A] = 10$, what is $\mathbb{E}[2X + Y/2|A]$?

- ◯ 14

- ◯ 18
- ◯ 9
- ◯ 13

(e) True or false: Chebyshev's Inequality can best be described as giving an upper bound on the distribution's right tail.

2. Tail bounds

Suppose $X \sim \mathsf{Binomial}(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds we've learned, and compare this to the true result.

(a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

(b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound.

(c) Give an upper bound for this probability using the Chernoff bound.

(d) Give the exact probability.

3. Exponential Tail Bounds

Let $X \sim \text{Exp}(\lambda)$ and $k > 1/\lambda$.

(a) Use Markov's inequality to bound $\text{P}(X \geq k)$.

(b) Use Markov's inequality to bound $\text{P}(X < k)$.

(c) Use Chebyshev's inequality to bound $\text{P}(X \geq k)$.

(d) What is the exact formula for $\text{P}(X \geq k)$?

(e) For $\lambda k \geq 3$, how do the bounds given in parts (a), (c), and (d) compare?

4. Robbie's Late!

Suppose the probability Robbie is late to teaching lecture on a given day is at most 0.01. Do not make any independence assumptions.

(a) Use a Union Bound to bound the probability that Robbie is late at least once over a 30-lecture quarter.

(b) Use a Union Bound to bound the probability that Robbie is **never** late over a 30-lecture quarter.

(c) Use a Union Bound to bound the probability that Robbie is late at least once over a 120-lecture quarter.