

Section 8: Solutions

Review of Main Concepts

- **Markov's Inequality:** Let X be a non-negative random variable, and $\alpha > 0$. Then,

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

- **Chebyshev's Inequality:** Suppose Y is a random variable with $\mathbb{E}[Y] = \mu$ and $\text{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$\mathbb{P}(|Y - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$$

- **(Multiplicative) Chernoff Bound:** Let X_1, X_2, \dots, X_n be independent Bernoulli random variables.

Let $X = \sum_{i=1}^n X_i$, and $\mu = \mathbb{E}[X]$. Then, for any $0 \leq \delta \leq 1$,

$$- \mathbb{P}(X \geq (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$$

$$- \mathbb{P}(X \leq (1 - \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$$

0. Central Limit Theorem Practice

You're playing ping pong with your friend, and want to keep playing until you've scored 15 points. Unfortunately, your friend is a much more skilled ping pong player than you, so you only win points 25% of the time (with each point being independent of the other points). Approximate the probability that you'll need to play at least 50 points before stopping. **Solution:**

Let X be the total number of points played. We want to approximate $\mathbb{P}(X \geq 50)$.

Let X_i be the number of points played starting after the $i - 1$ th point you win and up to and including the i th point you win, with X_1 the number of points up to and including the first point you win. Then, we have $X = \sum_{i=1}^{15} X_i$. Because you win each point independently with probability 0.25, we have $X_i \sim \text{Geo}(0.25)$. Thus,

$$\mathbb{E}[X_i] = 4$$

and

$$\text{Var}(X_i) = \frac{1 - 0.25}{(0.25)^2} = 12$$

Since X is the sum of 15 i.i.d. r.v.s with mean $\mu = 4$ and variance $\sigma^2 = 12$, by the central limit theorem we have $X \approx N \sim \mathcal{N}(15 \cdot 4, 15 \cdot 12)$. Thus,

$$\begin{aligned} \mathbb{P}(X \geq 50) &= \mathbb{P}(X \geq 49.5) && \text{Continuity correction} \\ &\approx \mathbb{P}(N \geq 49.5) && \text{CLT} \end{aligned}$$

Standardizing, we get the following:

$$\begin{aligned}\mathbb{P}(N \geq 49.5) &= \mathbb{P}\left(\frac{N - 15 \cdot 4}{\sqrt{15 \cdot 12}} \geq \frac{49.5 - 15 \cdot 4}{\sqrt{15 \cdot 12}}\right) \\ &\approx \mathbb{P}\left(\frac{N - 15 \cdot 4}{\sqrt{15 \cdot 12}} \geq -0.782\right) \\ &\approx \mathbb{P}(Z \geq -0.782) \\ &= 1 - \mathbb{P}(Z \leq -0.782) \\ &= 1 - \Phi(-0.782) \\ &= 1 - (1 - \Phi(0.782)) \\ &= \Phi(0.782) \\ &\approx 0.7823\end{aligned}$$

1. Content Review

- (a) True or false: the Union Bound always gives a result in $[0, 1]$.

Solution:

False. Consider X and Y , which are independent indicator random variables.

$$\text{Suppose } p_X(x) = \begin{cases} 0.75 & x = 0 \\ 0.25 & x = 1 \end{cases} \text{ and } p_Y(y) = \begin{cases} 0.75 & y = 0 \\ 0.25 & y = 1 \end{cases}.$$

Then we may apply the Union Bound to place a bound on $P(X = 0 \cup Y = 0)$:

$$P(X = 0 \cup Y = 0) \leq P(X = 0) + P(Y = 0) = 0.75 + 0.75 = 1.5.$$

In these cases, the Union Bound tells us very little, since the probability of any event occurring is at most 1.

- (b) True or false: Markov's Inequality always gives a non-negative result.

Solution:

True. Markov's Inequality is

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

as long as X is a non-negative random variable and $\alpha > 0$. Since X is a non-negative random variable, $\mathbb{E}[X] \geq 0$, so $\frac{\mathbb{E}[X]}{\alpha} \geq 0$.

- (c) Suppose C and D are discrete random variables. Then $\mathbb{E}[C|D = d] =$

- $\sum_d dp_{D|C}(d|c)$
- $\sum_c cp_{C|D}(c|d)$
- $\int_{-\infty}^{\infty} cf_{c|d}dx$
- $\frac{\mathbb{E}[C]}{\mathbb{E}[D]}$

Solution:

Choice b is the correct answer from the definition of conditional expectation for discrete random variables.

- (d) Suppose X and Y are random variables and A is an event. Given that $\mathbb{E}[X|A] = 4$ and $\mathbb{E}[Y|A] = 10$, what is $\mathbb{E}[2X + Y/2|A]$?

- 14
- 18
- 9
- 13

Solution:

Choice d is the correct answer since linearity of expectation still applies to conditional expectation:

$$\mathbb{E}[2X + Y/2|A] = \mathbb{E}[2X|A] + \mathbb{E}[Y/2|A] = 2\mathbb{E}[X|A] + \mathbb{E}[Y|A]/2 = 2 \cdot 4 + 10/2 = 8 + 5 = 13.$$

- (e) True or false: Chebyshev's Inequality can best be described as giving an upper bound on the distribution's right tail.

Solution:

False. Chebyshev's Inequality gives an upper bound on the sum of the probabilities of the left and right tails of the distribution.

2. Tail bounds

Suppose $X \sim \text{Binomial}(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds we've learned, and compare this to the true result.

- (a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

Solution:

We know that the expected value of a binomial distribution is np , so: $\mathbb{P}(X \geq 4) \leq \frac{\mathbb{E}[X]}{4} = \frac{2.4}{4} = 0.6$. We can use it since X is nonnegative.

- (b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound. **Solution:**

$\mathbb{P}(X \geq 4) = \mathbb{P}(X - 2.4 \geq 1.6) \leq \mathbb{P}(|X - 2.4| \geq 1.6)$ we can add those absolute value signs because that only adds more possible values, so it is an upper bound on the probability of $X - 2.4 \geq 1.6$. Then, using Chebyshev's inequality we get:

$$\mathbb{P}(|X - 2.4| \geq 1.6) \leq \frac{\text{Var}(X)}{1.6^2} = \frac{1.44}{1.6^2} = 0.5625$$

- (c) Give an upper bound for this probability using the Chernoff bound. **Solution:**

First, we solve for the values of δ that will allow us to use the Chernoff bound. We want $(1 + \delta)E[X] = (1 + \delta)2.4 = 4$. Solving for δ here gives use $\delta = \frac{2}{3}$. Now, we can directly plug into the Chernoff bound. $\mathbb{P}(X \geq 4) = \mathbb{P}(X \geq (1 + \frac{2}{3})2.4) \leq e^{-(\frac{2}{3})^2 \mathbb{E}[X]/3} = e^{-4 \times 2.4/27} \approx 0.7$

- (d) Give the exact probability. **Solution:**

Since X is a binomial, we know it has a range from 0 to n (or in this case 0 to 6). Thus, the possible values to satisfy $X \geq 4$ are 4, 5, or 6. We plug in the PMF for each to get: $\mathbb{P}(X \geq 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) = \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}0.4^6 \approx 0.1792$

3. Exponential Tail Bounds

Let $X \sim \text{Exp}(\lambda)$ and $k > 1/\lambda$.

(a) Use Markov's inequality to bound $\mathbb{P}(X \geq k)$.

Solution:

We can use Markov's inequality here because X is non-negative since it is an exponential distribution. We also know that $E[X] = 1/\lambda$ because $X \sim \text{Exp}(\lambda)$. By Markov's inequality, we get that:

$$\mathbb{P}(X \geq k) \leq \frac{1}{\lambda k}$$

(b) Use Markov's inequality to bound $\mathbb{P}(X < k)$. **Solution:**

From Markov's inequality (and our answer in (a)), we know that $\mathbb{P}(X \geq k) \leq \frac{1}{\lambda k}$. Then,

$$\begin{aligned} \mathbb{P}(X \geq k) &\leq \frac{1}{\lambda k} \\ -\mathbb{P}(X \geq k) &\geq -\frac{1}{\lambda k} && \text{multiplying by a negative flips the inequality} \\ 1 - \mathbb{P}(X \geq k) &\geq 1 - \frac{1}{\lambda k} \\ \mathbb{P}(X < k) &\geq 1 - \frac{1}{\lambda k} && \text{by definition of complement} \end{aligned}$$

Note that because we took the complement and the sign flipped, we have now found a *lower* bound for $\mathbb{P}(X < k)$.

(c) Use Chebyshev's inequality to bound $\mathbb{P}(X \geq k)$. **Solution:**

We rearrange algebraically to get into the form to apply Chebyshev's inequality. We then plug in the corresponding values and $\text{Var}(X) = \frac{1}{\lambda^2}$.

$$\mathbb{P}(X \geq k) = \mathbb{P}\left(X - \frac{1}{\lambda} \geq k - \frac{1}{\lambda}\right) \leq \mathbb{P}\left(\left|X - \frac{1}{\lambda}\right| \geq k - \frac{1}{\lambda}\right) \leq \frac{1}{\lambda^2(k - 1/\lambda)^2} = \frac{1}{(\lambda k - 1)^2}$$

(d) What is the exact formula for $P(X \geq k)$? **Solution:**

Using the CDF for an exponential distribution and definition of complement:

$$\mathbb{P}(X \geq k) = 1 - P(X \leq k) = 1 - (1 - e^{-\lambda k}) = e^{-\lambda k}$$

(e) For $\lambda k \geq 3$, how do the bounds given in parts (a), (c), and (d) compare?

Solution:

$$e^{-\lambda k} < \frac{1}{(\lambda k - 1)^2} < \frac{1}{\lambda k}$$

so Markov's inequality gives the worst bound.

4. Robbie's Late!

Suppose the probability Robbie is late to teaching lecture on a given day is at most 0.01. Do not make any independence assumptions.

(a) Use a Union Bound to bound the probability that Robbie is late at least once over a 30-lecture quarter. **So-**

lution:

Let R_i be the event Robbie is late to lecture on day i for $i = 1, \dots, 30$. Then, by the union bound,

$$\begin{aligned} \mathbb{P}(\text{late at least once}) &= \mathbb{P}\left(\bigcup_{i=1}^{30} R_i\right) \\ &\leq \sum_{i=1}^{30} \mathbb{P}(R_i) && \text{[union bound]} \\ &\leq \sum_{i=1}^{30} 0.01 && [\mathbb{P}(R_i) \leq 0.01] \\ &= 0.30 \end{aligned}$$

(b) Use a Union Bound to bound the probability that Robbie is **never** late over a 30-lecture quarter. **Solution:**

As in the previous part, let R_i be the event Robbie is late to lecture on day i for $i = 1, \dots, 30$. Then, by the union bound, we found that

$$\mathbb{P}(\text{late at least once}) \leq 0.30$$

The probability Robbie is never late is the complement of the probability he is late at least once over the 30 lectures. Taking the complement and doing algebra:

$$\begin{aligned} \mathbb{P}(\text{late at least once}) &\leq 0.30 \\ -\mathbb{P}(\text{late at least once}) &\geq -0.30 && \text{[multiplying by negative flips the inequality]} \\ 1 - \mathbb{P}(\text{late at least once}) &\geq 1 - 0.30 \\ \mathbb{P}(\text{never late}) &\geq 0.70 \end{aligned}$$

Note that we have now found a *lower* bound for this probability using the union bound because of taking the complement.

(c) Use a Union Bound to bound the probability that Robbie is late at least once over a 120-lecture quarter.

Solution:

Let R_i be the event Robbie is late to lecture on day i for $i = 1, \dots, 120$. Then, by the union bound,

$$\begin{aligned} \mathbb{P}(\text{late at least once}) &= \mathbb{P}\left(\bigcup_{i=1}^{120} R_i\right) \\ &\leq \sum_{i=1}^{120} \mathbb{P}(R_i) && \text{[union bound]} \\ &\leq \sum_{i=1}^{120} 0.01 && [\mathbb{P}(R_i) \leq 0.01] \\ &= 1.20 \end{aligned}$$

Notice that $\mathbb{P}(\text{late at least once}) \leq 1.20$ is not a very helpful bound since probabilities have to be at most 1 already.