CSE 312 Section 8

Tail Bounds

Announcements & Reminders

- HW6
  - Due on Wednesday 2/25
  - Late deadline Saturday 2/28 @ 11:59 pm

- HW7
  - Released
  - Due Wednesday 3/4 @ 11:59 pm
  - Late deadline Saturday 3/7 @ 11:59 pm

Review & Questions

Any lingering questions from this last week?

Each week in section, we'll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we'll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Review of Main Concepts

- **Markov's Inequality**: Let X be a non-negative random variable, and $\alpha > 0$. Then,

$$\mathbb{P}\left(X \geq \alpha\right) \leq \frac{\mathbb{E}\left[X\right]}{\alpha}$$

- **Chebyshev's Inequality**: Suppose Y is a random variable with $\mathbb{E}\left[Y\right] = \mu$ and $\text{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$\mathbb{P}\left(|Y - \mu| \geq \alpha\right) \leq \frac{\sigma^2}{\alpha^2}$$

- **(Multiplicative) Chernoff Bound**: Let $X_1, X_2, ..., X_n$ be *independent* Bernoulli random variables.

  Let $X = \Sigma_{i=1}^{n} X_i$, and $\mu = \mathbb{E}\left[X\right]$. Then, for any $0 \leq \delta \leq 1$,

  - $\mathbb{P}\left(X \geq (1+\delta)\,\mu\right) \leq e^{-\frac{\delta^2 \mu}{3}}$
  - $\mathbb{P}\left(X \leq (1-\delta)\,\mu\right) \leq e^{-\frac{\delta^2 \mu}{2}}$

Problem 2 – Tail Bounds

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Suppose $X \sim \text{Binomial}(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds we've learned, and compare this to the true result.

(a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

(b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound.

(c) Give an upper bound for this probability using the Chernoff bound.

(d) Give the exact probability.

Work on this with the people around you and then we'll go over it together!

Suppose $X \sim \mathsf{Binomial}(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds we've learned, and compare this to the true result.

(a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

We know that the expected value of a binomial distribution is np, so: $\mathbb{P}(X \geq 4) \leq \frac{\mathbb{E}[X]}{4} = \frac{2.4}{4} = 0.6$. We can use it since X is nonnegative.

(b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound.

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$\mathbb{P}(X \geq 4) = \mathbb{P}(X - 2.4 \geq 1.6) \leq \mathbb{P}(|X - 2.4| \geq 1.6)$ we can add those absolute value signs because that only adds more possible values, so it is an upper bound on the probability of $X - 2.4 \geq 1.6$. Then, using Chebyshev's inequality we get:

$\mathbb{P}(|X - 2.4| \geq 1.6) \leq \frac{Var(X)}{1.6^2} = \frac{1.44}{1.6^2} = 0.5625$

(c) Give an upper bound for this probability using the Chernoff bound.

First, we solve for the values of δ that will allow us to use the Chernoff bound. We want $(1+\delta)E[X] = (1+\delta)2.4 = 4$. Solving for δ here gives use $\delta = \frac{2}{3}$. Now, we can directly plug into the Chernoff bound.
$\mathbb{P}(X \geq 4) = \mathbb{P}(X \geq (1+\frac{2}{3})2.4) \leq e^{-(\frac{2}{3})^2\mathbb{E}[X]/3} = e^{-4\times 2.4/27} \approx 0.7$

(d) Give the exact probability.

Since X is a binomial, we know it has a range from 0 to n (or in this case 0 to 6). Thus, the possible values to satisfy $X \geq 4$ are 4, 5, or 6. We plug in the PMF for each to get: $\mathbb{P}(X \geq 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) = \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}0.4^6 \approx 0.1792$

Problem 0 –
Central Limit Theorem Practice

Problem 0 - Central Limit Theorem Practice

You're playing ping pong with your friend, and want to keep playing until you've scored 15 points. Unfortunately, your friend is a much more skilled ping pong player than you, so you only win points 25% of the time (with each point being independent of the other points). Approximate the probability that you'll need to play at least 50 points before stopping.

Work on this with the people around you and then we'll go over it together!

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You're playing ping pong with your friend, and want to keep playing until you've scored 15 points. Unfortunately, your friend is a much more skilled ping pong player than you, so you only win points 25% of the time (with each point being independent of the other points). Approximate the probability that you'll need to play at least 50 points before stopping.

Let X be the total number of points played before stopping. We want to approximate $P(X \geq 50)$.

For i in {1, …, 15}, Let X_i be the number of points played after the i - 1th point you win, up to and including the i-th point you win. Then, $X = \sum X_i$

Because you win each point independently with probability 0.25, $X_i \sim Geo(0.25)$. So, $E[X_i] = 4$, and $Var[X_i] = 12$

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Because you win each point independently with probability 0.25, $X_i \sim Geo(0.25)$. So, $E[X_i] = 4$, and $Var[X_i] = 12$.

By the central limit theorem, $X \approx N \sim Norm(15 \cdot 4, 15 \cdot 12)$.

Applying continuity correction, $P(X \geq 50) = P(X \geq 49.5) \approx P(N \geq 49.5)$

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Applying continuity correction, $P(X \geq 50) = P(X \geq 49.5) \approx P(N \geq 49.5)$

Standardizing N with $\mu = 60$ and $\sigma^2 = 180$, we get
$P(N \geq 49.5) = P((N - 60)/\sqrt{180} \geq (49.5 - 60) / \sqrt{180}) \approx P(Z \geq -0.783)$

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Standardizing N with $\mu = 60$ and $\sigma^2 = 180$, we get

$P(N \geq 49.5) = P((N - 60)/\sqrt{180} \geq (49.5 - 60) / \sqrt{180}) \approx P(Z \geq -0.783)$

Converting this to a Z-table form, we get

$P(Z \geq -0.783) = 1 - P(Z \leq -0.783) = 1 - \Phi(-0.783) = 1 - (1 - \Phi(0.783)) = \Phi(0.783) \approx 0.783$

So, $P(X \geq 50) \approx 0.783$

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Review Questions

Question #1

a) True or false: the Union Bound always gives a result in [0, 1]

- True
- False

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- True
- False

Question #2

b) True or False: Markov's Inequality always gives a non-negative result.

- True
- False

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- True
- False

Question #3

b) Suppose C and D are discrete random variables. Then $E[C|D = d] =$

- $\sum_d d \cdot p_{D|C}\,(d|c)$
- $\sum_c c \cdot p_{D|C}(d|c)$
- $\int_{-\infty}^{\infty} c f_{c|d} dx$
- $\frac{E[C]}{E[D]}$

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- $\sum_c c \cdot p_{D|C}(d|c)$
- $\int_{-\infty}^{\infty} c f_{c|d} dx$
- $\frac{E[C]}{E[D]}$

Question #4

c) Suppose X and Y are random variables and A is an event. Given that $E[X|A] = 4$ and $E[Y|A] = 10$, what is $E[2X + \frac{Y}{2}|A]$

- 14
- 18
- 9
- 13

Question #4

e) Suppose X and Y are random variables and A is an event. Given that $E[X|A] = 4$ and $E[Y|A] = 10$, what is $E[2X + \frac{Y}{2}|A]$

- 14
- 18
- 9
- 13

$$\mathbb{E}\left[2X + Y/2|A\right] = \mathbb{E}\left[2X|A\right] + \mathbb{E}\left[Y/2|A\right] = 2\mathbb{E}\left[X|A\right] + \mathbb{E}\left[Y|A\right]/2 = 2 \cdot 4 + 10/2 = 8 + 5 = 13.$$

Question #5

d) True or false: Chebyshev’s Inequality can best be described as giving an upper bound on the distribution’s right tail.

- True
- False

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- True
- False

Problem 3 – Exponential Tail Bounds

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Let $X \sim \mathrm{Exp}(\lambda)$ and $k > \frac{1}{\lambda}$.

a) Use Markov's inequality to bound $P(X \geq k)$.
b) Use Markov's inequality to bound $P(X < k)$.
c) Use Chebyshev's inequality to bound $P(X \geq k)$.
d) What is the exact formula for $P(X \geq k)$.
e) For $\lambda k \geq 3$, how do the bounds given in parts a, c, and d compare?

Work on this with the people around you and then we'll go over it together!

Problem 3 – Exponential Tail Bounds

a) Use Markov's inequality to bound $P(X \geq k)$.

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a) Use Markov's inequality to bound $P(X \geq k)$.

We can use Markov's inequality here because X is non-negative since it is an exponential distribution. We also know that $E[X] = \lambda$ because $X \sim \text{Exp}(\lambda)$. By Markov's inequality, we get that:

$$\mathbb{P}(X \geq k) \leq \frac{1}{\lambda k}$$

Problem 3 – Exponential Tail Bounds

b) Use Markov's inequality to bound $P(X < k)$.

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b) Use Markov's inequality to bound $P(X < k)$.

From Markov's inequality (and our answer in (a)), we know that $P(X \geq k) \leq \frac{1}{\lambda k}$. Then,

$$\begin{aligned} P(X \geq k) &\leq \frac{1}{\lambda k} \\ -P(X \geq k) &\geq -\frac{1}{\lambda k} && \text{multiplying be a negative flips the inequality} \\ 1 - P(X \geq k) &\geq 1 - \frac{1}{\lambda k} \\ P(X < k) &\geq 1 - \frac{1}{\lambda k} && \text{by definition of complement} \end{aligned}$$

Note that because we took the complement and the sign flipped, we have now found a *lower* bound for $P(X < k)$.

Problem 3 – Exponential Tail Bounds

c) Use Chebyshev's inequality to bound $P(X \geq k)$.

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c) Use Chebyshev's inequality to bound $P(X \geq k)$.

We rearrange algebraically to get into the form to apply Chebyshev's inequality. We then plug in the corresponding values and $Var(X) = \frac{1}{\lambda^2}$.

$$\mathbb{P}(X \geq k) = \mathbb{P}\left(X - \frac{1}{\lambda} \geq k - \frac{1}{\lambda}\right) \leq \mathbb{P}\left(\left|X - \frac{1}{\lambda}\right| \geq k - \frac{1}{\lambda}\right) \leq \frac{1}{\lambda^2(k - 1/\lambda)^2} = \frac{1}{(\lambda k - 1)^2}$$

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d) What is the exact formula for $P(X \geq k)$.

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d) What is the exact formula for $P(X \geq k)$.

Using the CDF for an exponential distribution and definition of complement:

$$\mathbb{P}(X \geq k) = 1 - P(X \leq k) = 1 - (1 - e^{-\lambda k}) = e^{-\lambda k}$$

Problem 3 – Exponential Tail Bounds

e) For $\lambda k \geq 3$, how do the bounds given in parts a, c, and d compare?

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e) For $\lambda k \geq 3$, how do the bounds given in parts a, c, and d compare?

$$e^{-\lambda k} < \frac{1}{(\lambda k - 1)^2} < \frac{1}{\lambda k}$$

so Markov's inequality gives the worst bound.

That’s All, Folks!

Thanks for coming to section this week!
Any questions?