Section 2: Solutions

Review of Main Concepts (Counting)

- **Binomial Theorem:** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  In general: +singles - doubles + triples - quads + ...
- **Pigeonhole Principle:** If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, $\lceil \frac{n}{k} \rceil$) pigeons.
- **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: (1) find the total number of ways to do everything and then (2) subtract the number of ways to *not* do X.
- **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S
- **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
- **Union:** The union of two events E and F is denoted $E \cup F$
- **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
- **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$
- **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
- **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
- **Probability of an event E:** denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$

Axioms of Probability and their Consequences

(a) **(Non-negativity)** For any event E, $\mathbb{P}(E) \geq 0$

(b) **(Normalization)** $\mathbb{P}(\Omega) = 1$

(c) **(Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

- **Corollaries of these axioms:**
  - $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$
  - If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$
  - $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$
- **Equally Likely Outcomes:** If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \dfrac{|E|}{|\Omega|}$.
  - Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

1. Content Review

(a) **True** or **False**. The following statement is always true: $|A \cup B| = |A| + |B|$ **Solution:**

False. Unless A and B do not overlap, by the property of inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B|$.

(b) If there are 7 pigeons that each go into one of 3 holes:

◯ There is at least one hole with exactly 3 pigeons in it.

◯ There is at least one hole with at least 3 pigeons in it.

◯ There is exactly one hole with at least 3 pigeons in it.

Solution:

By the pigeonhole principle, there will be *at least* one hole with *at least* $\lceil \frac{7}{3} \rceil = 3$ pigeons in it.

(c) $(x+y)^n =$

◯ $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

◯ $\sum_{k=0}^{n} x^k y^{n-k}$

◯ $\sum_{k=0}^{n} \binom{n}{k} x^k$

Solution:

(a) by definition of binomial theorem.

(d) An **event** and **sample space** are, respectively:

◯ The total set of possible outcomes; A subset of the event space

◯ A subset of the sample space; The total set of possible outcomes

◯ Some set of outcomes; Any other set of outcomes.

Solution:

(b) by definition of event and sample space. An event is always a subset of the sample space.

(e) **True** or **False**. It is always true that $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$. **Solution:**

False. This is only true if all outcomes in the sample space are *equally likely*.

(f) If A is the event that I eat an apple today,

◯ $\overline{A}$ is the event that I eat a banana today, and $P(A) + P(\overline{A}) = 0.5$

◯ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 0$

◯ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 1$

Solution:

(c) is correct. $\overline{A}$ is the *complement* of the event A. In this case, that is the event that I do *not* eat an apple today. $P(A) + P(\overline{A}) = 1$ since I will definitely either eat or not eat an apple. This property holds for all events!

(g) **True** or **False**. For any two events A and B $P(A \cup B) > P(A) + P(B)$. **Solution:**

False. The correct statement would be $P(A \cup B) \leq P(A) + P(B)$. If A and B are mutually exclusive (i.e., $A \cap B$ is an empty set), then $P(A \cup B) = P(A) + P(B)$. Another way to think about it is, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (this is similar to the property of inclusion-exclusion!). Sometimes, $P(A \cap B)$ might be 0 when A and B can't happen at the same time, so then $P(A \cup B)$ would equal $P(A) + P(B)$

2. A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members). **Solution:**

Remember that a combinatorial proof just requires that we show both sides are equivalent ways of counting a situation.

Left hand side: Choose a team of size r and a captain for that team (from among the r) by first choosing the captain (n choices) and then choosing the rest of the team $\binom{n-1}{r-1}$.

Right hand side: Choose a team of size r and a captain for that team by first choosing the team ($\binom{n}{r}$ choices) and then choosing the captain from among the members of the team (r choices).

3. Subsubset

Let $[n] = \{1, 2, ..., n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A, B) are there such that $A \subseteq B \subseteq [n]$?

Solution:

There are two ways to do this question:

First way: Apply the sum rule by adding up the number of ways of doing this where B has size k, where k is any integer between 0 and n. Now apply the product rule to find the number of ways to choose B of size exactly k (there are $\binom{n}{k}$ possibilities for B), and then once B is selected, count the number of ways of choosing A which has to be a subset of B (2^k ways). Hence the number of such ordered pairs of subsets is

$$\sum_{k=0}^{n}\binom{n}{k}2^k = \sum_{k=0}^{n}\binom{n}{k}2^k1^{n-k} = 3^n$$

by the Binomial Theorem.

Second way: Realize that, if there are no restrictions, for each element i of $1, ..., n$, there are four possibilities: it can be in only A, only B, both, or neither. In our case, there is only one that is not valid (violates $A \subseteq B$):

being in A but not B. Hence there are 3 choices for each element, so the total number of such ordered pairs of subsets is 3^n.

4. GREED INNIT

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

Solution:

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \setminus (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and by complementing, $|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

Repeat the question for the letters "AAAAABBB". **Solution:**

Note that no A's and no B's can be adjacent. So let us put the B's down first: $\_B\_B\_B\_$

By the pigeonhole principle, two A's must go in the same slot, but then they would be adjacent, so there are $\boxed{\text{no ways}}$.

5. Friendships

Show that in any group n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

(a) Everyone has at least one friend. **Solution:**

Everyone has between 1 and $n-1$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

(b) At least one person has no friends. **Solution:**

Here, we need to observe that if someone has 0 friends, then nobody has $n-1$ friends (by the symmetry of the friendship relation). Then, possible choices are now between 0 and $n-2$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

6. Powers and divisibility

Prove that there exist two powers of 7 whose difference is divisible by 2003. (You may want to use the Pigeonhole principle.)

Solution:

Recall the division theorem, which states that for any integer a, there exists *unique integers* q, r such that

$$a = 2003q + r \ ,$$

where $0 \leq r < 2003$. Notably, there are 2003 possible remainders (values that r can take). To that end, consider a sequence of powers

$$7^0 \quad , \quad 7^1 \quad , \quad 7^2 \quad , \quad 7^3 \quad , \quad \dots \quad , \quad 7^{2003} \ .$$

The above sequence has 2004 elements. For the i^{th} member of the sequence, i.e., 7^i for $0 \leq i \leq 2003$, we can express it as $2003q_i + r_i$, where q_i and r_i are its respective quotient and remainder after dividing by 2003. There are 2004 such r_is (pigeons) which need to be mapped to 2003 possible remainders (pigeonholes). By the pigeonhole principle, there must exist a pair, say 7^n and 7^m, (where $n > m$) such that $r_n = r_m$. Then

$$7^n - 7^m = (2003q_n + r_n) - (2003q_m + r_m) = 2003(q_n - q_m) \ ,$$

which is divisible by 2003.

It may be simpler to understand this with the help of a smaller example. Lets use 4 instead of 2003. We would then construct our sequence as

$$1 = 7^0 \quad , \quad 7 = 7^1 \quad , \quad 49 = 7^2 \quad , \quad 343 = 7^3 \quad , \quad 2401 = 7^4 \ .$$

The remainders after dividing by 4 would then be $1, 3, 1, 3, 1$. Lets take 7^1 and 7^3 as our 7^m and 7^n respectively. Then we find $7^3 - 7^1 = 343 - 7 = 336$. 336 is divisible by 4.

7. Dinner Party

At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place. **Solution:**

For $i = 1, ..., n$, let r_i be the number of rotations clockwise needed for the i^{th} person to be in their spot. Each r_i can be between 1 and $n - 1$ (not 0 since no one is at their nametag, and not n since it is equivalent to 0). Since there are n people and only $n - 1$ possible values for the rotations, at least two must have the same value by the pigeonhole principle. Rotate the table clockwise by that much, and at least two people will be in the correct place.

8. Count the Solutions

Consider the following equation: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the equation?

Solution:

(Hint: Think about splitting a sequence of 70 1's into 6 blocks, each block consisting of consecutive 1's in the sequence. The number of 1's in the i-th block corresponds to the value of a_i. Note that the i-th block is allowed to be empty, corresponding to $a_i = 0$.)

Using the stars and bars method, we get:

$$\binom{70+6-1}{6-1} = \binom{75}{5} = 17,259,390$$

9. Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. Compute $\mathbb{P}(E)$, where E is the event that the suits of the shuffled cards are in alternating order.

Solution:

The sample space Ω is all re-orderings possible: there are $|\Omega| = 6!$ such. Now for E, order the spades and hearts independently, so there are $3!^2$ ways to do so. Finally choose whether you want hearts or spades first. All such orderings are equally likely, so $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{2\cdot 3!^2}{6!}$.

10. Balls from an Urn

Say an urn contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

(a) Give a probability space describing the experiment. **Solution:**

$\Omega = \{B, R, G\}^2$ and $\mathbb{P}(\omega) = 1/9$ for all $\omega \in \Omega$.

(b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.) **Solution:**

The event is $A = \{RR\}$. Its probability is $\mathbb{P}(A) = \frac{|A|}{9} = \frac{1}{9}$.

(c) What is the probability that at most one ball is red? **Solution:**

This is just A^c, the complement of A. We know that $\mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 1 - \frac{1}{9} = 8/9$.

(d) What is the probability that we get at least one green ball? **Solution:**

This is the event $B = \{GR, GB, GG, RG, BG\}$, and thus $\mathbb{P}(B) = \frac{|B|}{9} = \frac{5}{9}$.

(e) Repeat **b)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn. **Solution:**

Here, the probability space changes: First of all, the outcomes RR, GG, BB are not possible any more, so let us remove them from Ω, which is now $\Omega = \{BG, BR, GB, GR, RB, RG\}$. Note that now we have

$\mathbb{P}(\omega) = 1/3 \cdot 1/2 = 1/6$ for every outcome, because we have three choices for the first ball, but only two for the second. Equivalently, we could note that 6 elements remain in Ω and they have equal probabilities, so the probability of each must be $1/6$.

Continuing to **b)**, it can never be that both balls are red. Therefore, for **b)**, the probability becomes 0. Thus **c)** becomes 1 (i.e., the associated event *is* Ω.)

For **d)**, the event becomes $B = \{GR, GB, RG, BG\}$, and $\mathbb{P}(B) = 4 \cdot \frac{1}{6} = \frac{2}{3}$.

11. Weighted Die

Consider a weighted die such that

- $\mathbb{P}(1) = \mathbb{P}(2)$,
- $\mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6)$, and
- $\mathbb{P}(1) = 3 \cdot \mathbb{P}(3)$.

What is the probability that the outcome is 3 or 4?

Solution:

By the second axiom of probability, the sum of probabilities for the sample space must equal 1. That is, $\sum_{i=1}^{6} \mathbb{P}(i) = 1$. Since $\mathbb{P}(1) = \mathbb{P}(2)$ and $\mathbb{P}(1) = 3\mathbb{P}(3)$, we have that: $1 = \mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) + \mathbb{P}(4) + \mathbb{P}(5) + \mathbb{P}(6) = 3 \cdot \mathbb{P}(3) + 3 \cdot \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) = 10 \cdot \mathbb{P}(3)$

Thus, solving algebraically, $\mathbb{P}(3) = 0.1$, so $\mathbb{P}(3) = \mathbb{P}(4) = 0.1$. Since rolling a 3 and 4 are disjoint events, then $\mathbb{P}(3 \text{ or } 4) = \mathbb{P}(3) + \mathbb{P}(4) = 0.1 + 0.1 = 0.2$.

12. Shuffling Cards

We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace $>$ King $>$ Queen $>$ Jack $>$ 10 $> \cdots >$ 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one? **Solution:**

First off, the sample space Ω here consists of all pairs of cards – which we can represent by their value *and* suit, e.g., $(4\clubsuit, A\diamondsuit)$. There $52 \cdot 51 = 2652$ possible outcomes, therefore $\mathbb{P}(\omega) = \frac{1}{2652}$ for all $\omega \in \Omega$.

Let us now look at the size of the event E containing all pairs where the first card is strictly larger than the second. Then, the number of pairs of values of cards a and b where $a < b$ is exactly $\binom{13}{2} = 13 \cdot 6 = 78$. We can then assign suits to each of them – given the cards are different, all suits are possible for each, so there are $4^2 = 16$ choices. Thus, overall,

$$|E| = 16 \cdot 78 = 1248 \; .$$

Therefore,

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{16 \cdot 78}{52 \cdot 51} = \frac{13 \cdot 3 \cdot 2^5}{13 \cdot 3 \cdot 2^2 \cdot 17} = \frac{8}{17} \approx 0.47 \; .$$

13. Flipping Coins

Note: The content in this problem (conditional probability) will be covered in lecture on Friday! A coin is tossed twice. The coin is “heads” one quarter of the time. You can assume that the second toss is independent of the first toss.

(a) What is the probability that the second toss is “heads” given that the first toss is “tails”? **Solution:**

Consider the probability space with sample space $\Omega = \{HH, TT, HT, TH\}$. Because heads come $1/4$ of the time, and tails $3/4$, we have $\mathbb{P}(HH) = 1/4 \times 1/4 = 1/16$, $\mathbb{P}(HT) = \mathbb{P}(TH) = 3/4 \times 1/4 = 3/16$ and finally $\mathbb{P}(TT) = 9/16$.

Then, let A be the event that the first coin is tails, and let B be the event that the second coin is heads. Then,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} .$$

Note that $A = \{TT, TH\}$ and $B = \{HH, TH\}$, and thus

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(TT) + \mathbb{P}(TH) = 9/16 + 3/16 = 12/16 = 3/4 \\ \mathbb{P}(A \cap B) &= \mathbb{P}(TH) = 3/16 . \end{aligned}$$

Therefore, $\mathbb{P}(B|A) = (3/16)/(3/4) = 1/4$.

It is important to realize that this exactly what we would have expected – indeed, we model the coins to be independent.

(b) What is the probability that the second toss is “heads” given that at least one of the tosses is “tails”? **Solution:**

Let A, B be the same events as in **a)**. We define $C = \{TH, TT, TT\}$, and we want $\mathbb{P}(B|C)$. Note that

$$\begin{aligned} \mathbb{P}(C) &= 1 - \mathbb{P}(HH) = 15/16 \\ \mathbb{P}(B \cap C) &= \mathbb{P}(TH) = 3/16 . \end{aligned}$$

Therefore,

$$\mathbb{P}(B \mid C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{3/16}{15/16} = \frac{3}{15} = \frac{1}{5} .$$

(c) In the probability space of this task, give an example of two events that are disjoint but not independent. **Solution:**

$E_1 = \{TT\}$ and $E_2 = \{HH\}$ are disjoint, but not independent. Indeed, $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(\emptyset) = 0$, but each event occurs with positive probability, and so $\mathbb{P}(E_1) \cdot \mathbb{P}(E_2) > 0$.

(d) In the probability space of this task, give an example of two events that are independent but not disjoint. **Solution:**

$E_1 = \{TH, HH\}$ and $E_2 = \{TH, TT\}$ are not disjoint, but are independent.

14. Balls from an Urn – Take 2

Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

(a) What is the probability that all three balls are all of the same color? **Solution:**

The experiment is modeled with $\Omega = \{r, b\}^3$. Probabilities are assigned as we have seen in class, by assuming every draw is uniform among the remaining balls. Then, note that $\mathbb{P}(rrr) = 3/7 \cdot 2/6 \cdot 1/5 = 1/35$ and $\mathbb{P}(bbb) = 4/7 \cdot 3/6 \cdot 2/5 = 4/35$. Therefore, the probability that they all have the same color is $1/35 + 4/35 = 1/7$.

(b) What is the probability that we get more than one red ball given the first ball is red? **Note: The content in this problem (conditional probability) will be covered in lecture on Friday!**

Solution:

Let R be the event that the first ball is red. Since we select the first ball uniformly, $\mathbb{P}(R) = \frac{3}{7}$. (This can be computed explicitly from Ω.) We also consider the event M that we have more than one red ball. Let M be the event that more than one ball is red. We need to now compute the probability $\mathbb{P}(M \cap R)$, but note that by the law of total probability

$$\mathbb{P}(M \cap R) = \mathbb{P}(R) - \mathbb{P}(M^c \cap R) = 3/7 - \mathbb{P}(M^c \cap R) \ .$$

We could compute this probability directly from Ω, but there is an easier way. Note that $M^c \cap R$ is the event that the first ball is red, and both remaining balls are blue. In particular,

$$\mathbb{P}(M^c \cap R) = \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} = \frac{6}{35} \ .$$

Thus, $\mathbb{P}(M \cap R) = 3/7 - 6/35 = 9/35$, and

$$\mathbb{P}(M|R) = \frac{\mathbb{P}(M \cap R)}{\mathbb{P}(R)} = \frac{9/35}{3/7} = \frac{3}{5} \ .$$

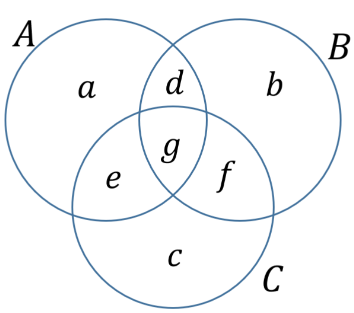
15. Additivity of Probability

Use the additivity of probability to prove that

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C) \ .$$

Solution:

We can now compute the probabilities of the events associate with the 7 areas of the Venn Diagram:



$$\begin{aligned} a &= \mathbb{P}(A \setminus (B \cup C)) \\ b &= \mathbb{P}(B \setminus (A \cup C)) \\ c &= \mathbb{P}(C \setminus (A \cup B)) \\ d &= \mathbb{P}((A \cap B) \setminus C) \\ e &= \mathbb{P}((A \cap C) \setminus B) \\ f &= \mathbb{P}((B \cap C \setminus A) \\ g &= \mathbb{P}(A \cap B \cap C) \end{aligned}$$

By the additivity of probability

$$\mathbb{P}(A \cup B \cup C) = a + b + c + d + e + f + g \ .$$

Also,

$$d + g = \mathbb{P}(A \cap B) \ , \quad e + g = \mathbb{P}(A \cap C) \ , \quad f + g = \mathbb{P}(B \cap C) \ ,$$

and

$$a + d + e + g = \mathbb{P}(A) \ , \quad b + d + f + g = \mathbb{P}(B) \ , \quad c + e + f + g = \mathbb{P}(C) \ .$$

Now, we just verify that

$$\begin{aligned}(a + d + e + g) &+ (b + d + f + g) + (c + e + f + g) - (d + g) - (e + g) - (f + g) + g \\ &= a + b + c + 2d + 2e + 2f + 3g - d - e - f - 3g + g = a + b + c + d + e + f + g \ .\end{aligned}$$