CSE 312 Section 2

Intro to Discrete Probability

Administrivia

Announcements & Reminders

- Office Hours
  - Offered every day! Mostly in-person, a few zoom options
  - Times posted on the calendar on the course website

- HW1
  - Was due yesterday, Wednesday 1/15 @ 11:59pm
  - Late deadline Saturday 1/18 @ 11:59pm (max of 3 late days per homework)

- HW2
  - Released on the course website
  - Due Wednesday 1/22 @ 11:59pm

Homework

- Submissions
  - LaTeX (highly encouraged)
    - overleaf.com
    - Many TAs are familiar with Overleaf and using LaTeX
  - Word Editor that supports mathematical equations
  - Handwritten neatly and scanned
- Homework will typically be due on Wednesdays at 11:59pm on Gradescope
- Each assignment can be submitted a max of 72 hours late
- You have **6 late days total** to use throughout the quarter
  - Anything beyond that will result in a deduction on further late assignments

Feeling Stuck?

- Many of the strategies you need to solve homework problems are usually **practiced in section**
- For Homework 2, make sure to review Problems 2, 6, and 7 on the section handout today to practice combinatorics on the Pigeonhole Principle
- There are many ways to get help! EdStem, office hours, lecture slides, etc.

Review

Any lingering questions from this last week?

Problem 7 - Dinner Party

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At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

Work on this problem with the people around you, and then we'll go over it together!

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For $i = 1, ..., n$, let r_i be the number of rotations clockwise needed for the ith person to be in their spot.

Each r_i can be between 1 and n − 1 (not 0 since no one is at their nametag, and not n since it is equivalent to 0). Since there are n people and only $n - 1$ possible values for the rotations, at least two must have the same value **by the pigeonhole principle.**

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Rotate the table clockwise by that much, and at least two people will be in the correct place!

Problem 10 – Balls from an Urn

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Say an urn contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls with replacement, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.
b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)
c) What is the probability that at most one ball is red?
d) What is the probability that we get at least one green ball?
e) Repeat a)-d) for the case where the balls are drawn without replacement, i.e., when the first ball is drawn, it is not placed back from the urn.

Work on this problem with the people around you, and then we'll go over it together!

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a) Give a probability space describing the experiment.

$\Omega = \{BB, BR, BG, RB, RR, RG, GB, GR, GG\} = \{B, R, G\}^2$

$\mathbb{P}(\omega) = 1/9$ for all $\omega \in \Omega$

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The probability of the event is $\mathbb{P}(B) = \frac{|B|}{9} = 1/9$

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The event is $C = \{BB, BR, BG, RB, RG, GB, GR, GG\}$

The probability of the event is $\mathbb{P}(C) = \frac{|C|}{9} = 8/9$

Alternatively, this is just B^C, the complement of B.

We know that $\mathbb{P}(B^C) = 1 - \mathbb{P}(B) = 1 - 1/9 = 8/9$

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The event is $D = \{BG, GB, RG, GR, GG\}$

The probability of the event is $\mathbb{P}(D) = \frac{|D|}{9} = 5/9$

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e) Repeat b)-d) for the case where **the balls are drawn without replacement.**

a) Does the probability space change? If so, what is it now?

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$\Omega = \{BR, BG, RB, RG, GB, GR\}$

$\mathbb{P}(\omega) = \frac{1}{3} \cdot \frac{1}{2} = 1/6$ for all $\omega \in \Omega$ (3 choices for the first ball, 2 for the second)

b) What is the probability that both balls are red?

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b) What is the probability that both balls are red?

$B = \{\}$ We can't have both be red, so $\mathbb{P}(B) = 0$

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$D = \{BG, GB, RG, GR\}$ so $\mathbb{P}(D) = \frac{4}{6} = \frac{2}{3}$

Problem 12 – Shuffling Cards

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We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > · · · > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely). What is the probability the first card on the deck is (strictly) larger than the second one?

Work on this problem with the people around you, and then we'll go over it together!

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First off, the sample space Ω here consists of all pairs of cards – which we can represent by their value and suit, e.g., $(4\clubsuit, A\diamondsuit)$. There are $52 \cdot 51 = 2652$ possible outcomes, therefore $\mathbb{P}(\omega) = \frac{1}{2652}$ for all $\omega \in \Omega$.

Let us now look at the size of the event E containing all pairs where the first card is strictly larger than the second. Then, the number of pairs of values of cards a and b where $a < b$ is exactly $\binom{13}{2} = 13 \cdot 6 = 78$. We can then assign suits to each of them – given the cards are different, all suits are possible for each, so there are $4^2 = 16$ choices. Thus, overall,

$$|E| = 16 \cdot 78 = 1248$$

Therefore, $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{16 \cdot 78}{52 \cdot 51} = \frac{13 \cdot 3 \cdot 2^5}{13 \cdot 3 \cdot 2^2 \cdot 17} = \frac{8}{17} \approx 0.47$

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Consider a weighted die such that

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- $\mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6)$
- $\mathbb{P}(1) = 3 \cdot \mathbb{P}(3)$

What is the probability that the outcome is 3 or 4?

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$$
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&= 3 \cdot \mathbb{P}(3) + 3 \cdot \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) = 10 \cdot \mathbb{P}(3)
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Thus, solving algebraically, $\mathbb{P}(3) = 0.1$, so $\mathbb{P}(3) = \mathbb{P}(4) = 0.1$.

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Thus, solving algebraically, $\mathbb{P}(3) = 0.1$, so $\mathbb{P}(3) = \mathbb{P}(4) = 0.1$.

Since rolling a 3 and 4 are disjoint events, then $\mathbb{P}(3 \text{ or } 4) = \mathbb{P}(3) + \mathbb{P}(4) = 0.1 + 0.1 = 0.2$

Problem 2 - Subsubset

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- Let $[n] = \{1,2,\ldots,n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A, B) are there such that $A \subseteq B \subseteq [n]$?

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Realize that, if there are no restrictions, for each element i of $1, \dots, n$, there are four possibilities: it can be in only A, only B, both, or neither. In our case, there is only one that is not valid (violates $A \subseteq B$): being in A but not B. Hence there are 3 choices for each element, so the total number of such ordered pairs of subsets is

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$$3^n$$

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Alternately, apply the sum rule by adding up the number of ways of doing this where B has size k, where k is any integer between 0 and n. Now apply the product rule to find the number of ways to choose B of size exactly k (there are $\binom{n}{k}$ possibilities for B), and then once B is selected, count the number of ways of choosing A which has to be a subset of B (2^k ways).

Hence, by the Binomial Theorem, the number of such ordered pairs of subsets is

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$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = 3^n$$

Problem 4 – GREED INNIT

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Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other. For example, “INGREEDINT” is invalid because the two E’s are adjacent.

Repeat the question for the letters “AAAAABBB”.

Work on this problem with the people around you, and then we’ll go over it together!

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By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ = singles – doubles + triples, and by complementing, $|\Omega \backslash (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

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Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other.

Singles = $|A_I| + |A_E| + |A_N|$
Doubles = $|A_I \cap A_E| + |A_I \cap A_N| + |A_E \cap A_N|$
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First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N’s (multinomial coefficient).

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First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N’s (multinomial coefficient).

$|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I’s as one entity.
Clearly, $|A_I| = |A_E| = |A_N|$, so Singles = $3 \cdot \frac{9!}{2!2!}$

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Clearly, $|A_I| = |A_E| = |A_N|$, so Singles = $3 \cdot \frac{9!}{2!2!}$
$|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity. $|A_I \cap A_E| = |A_I \cap A_N| = |A_E \cap A_N|$, so Doubles = $3 \cdot \frac{8!}{2!}$

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Singles = $|A_I| + |A_E| + |A_N|$
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Triples = $|A_I \cap A_E \cap A_N|$

First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N’s (multinomial coefficient).

$|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I’s as one entity.
Clearly, $|A_I| = |A_E| = |A_N|$, so Singles = $3 \cdot \frac{9!}{2!2!}$
$|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I’s as one entity and the two adjacent E’s as one entity. $|A_I \cap A_E| = |A_I \cap A_N| = |A_E \cap A_N|$, so Doubles = $3 \cdot \frac{8!}{2!}$
Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I’s, E’s, and N’s as one entity.

4 – GREED INNIT

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other.

Singles = $|A_I| + |A_E| + |A_N|$
Doubles = $|A_I \cap A_E| + |A_I \cap A_N| + |A_E \cap A_N|$
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Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\frac{10!}{2!2!2!} - \left(3 \cdot \frac{9!}{2!2!} - 3 \cdot \frac{8!}{2!} + 7!\right)$

4 – GREED INNIT

Repeat the question for the letters “AAAAABBB”.

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Repeat the question for the letters “AAAAABBB”.

Note that no A’s and no B’s can be adjacent. So let us put the B’s down first:

_B_B_B_

By the pigeonhole principle, two A’s must go in the same slot, but then they would be adjacent, so there are no ways .

Problem 5 - Friendships

5 – Friendships

- Show that in any group n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.) Solve in particular the following two cases individually:

a) Everyone has at least one friend.

b) At least one person has no friends.

Work on this problem with the people around you, and then we'll go over it together!

5 – Friendships

a) Everyone has at least one friend.

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Everyone has between 1 and $n-1$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

5 – Friendships

b) At least one person has no friends.

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Here, we need to observe that if someone has 0 friends, then nobody has $n-1$ friends (by the symmetry of the friendship relation). Then, possible choices are now between 0 and $n-2$ friends (i.e., $n-1$ holes), and there are n people (the “pigeons”). Therefore, two of them will have the same number of friends.

That’s All, Folks!

Thanks for coming to section this week!
Any questions?

Problem 6 – Powers and Divisibility

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Prove that there exist two powers of 7 whose difference is divisible by 2003. (You may want to use the Pigeonhole principle.)

Work on this problem with the people around you, and then we'll go over it together!

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Prove that there exist two powers of 7 whose difference is divisible by 2003. (You may want to use the Pigeonhole principle.)

HINT: think about what are the pigeons, and what are the pigeonholes?

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Therefore, by Pigeonhole principle, there exists a pair of members, say 7^n and 7^m, where $n > m$, such that $r_n = r_m$.

Then their difference $7^n - 7^m = 2003x_n + r_n - 2003x_m - r_m = 2003(x_n - x_m)$ is divisible by 2003.