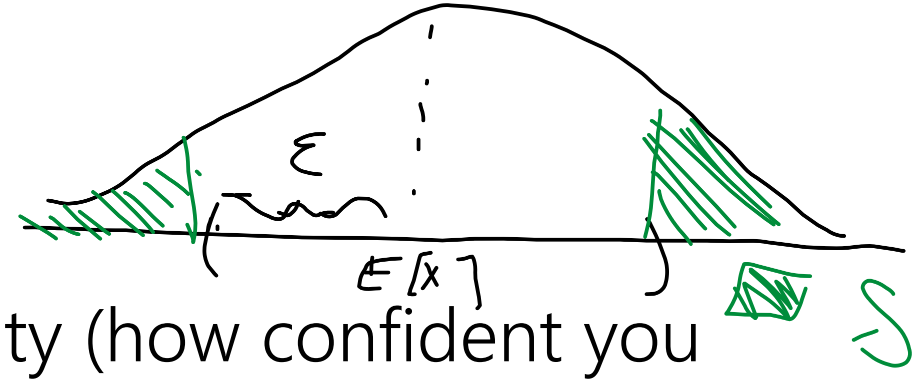
Tail Bounds

CSE 312 Winter 26
Lecture 19

Confidence Intervals



A "confidence interval" tells you the probability (how confident you should be) that your random variable fell in a certain range (interval)

Usually "close to its expected value"

$$\mathbb{P}(|X - \mu| > \varepsilon) \leq \delta$$

equivalently $\mathbb{P}(|X - \mu| \leq \varepsilon) > 1 - \delta$

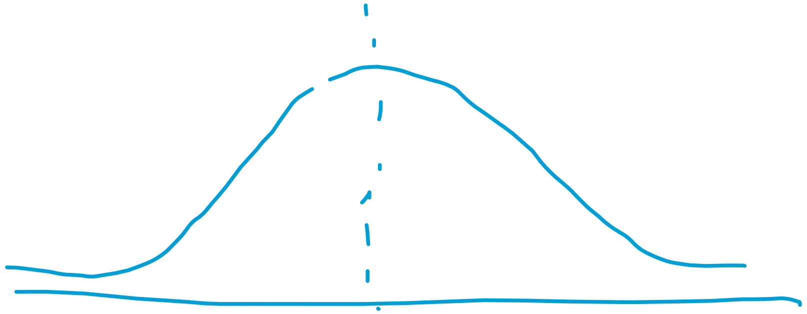
If your RV has expectation equal to the value you're searching for (like our polling example) you get a probability of being "close enough" to the target value.

What's a Tail Bound?

When we were finding our margin of error, we didn't need an exact calculation of the probability.

We needed an inequality: the probability of being outside the margin of error was **at most** 5%.

A tail bound (or concentration inequality) is a statement that bounds the probability in the "tails" of the distribution (says there's very little probability far from the center) or (equivalently) says that the probability is concentrated near the expectation.



Our First bound

Two statements are equivalent.
Left form is often easier to use.
Right form is more intuitive.

Markov's Inequality

Let X be a random variable supported (only) on non-negative numbers. For any $t > 0$

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

Markov's Inequality

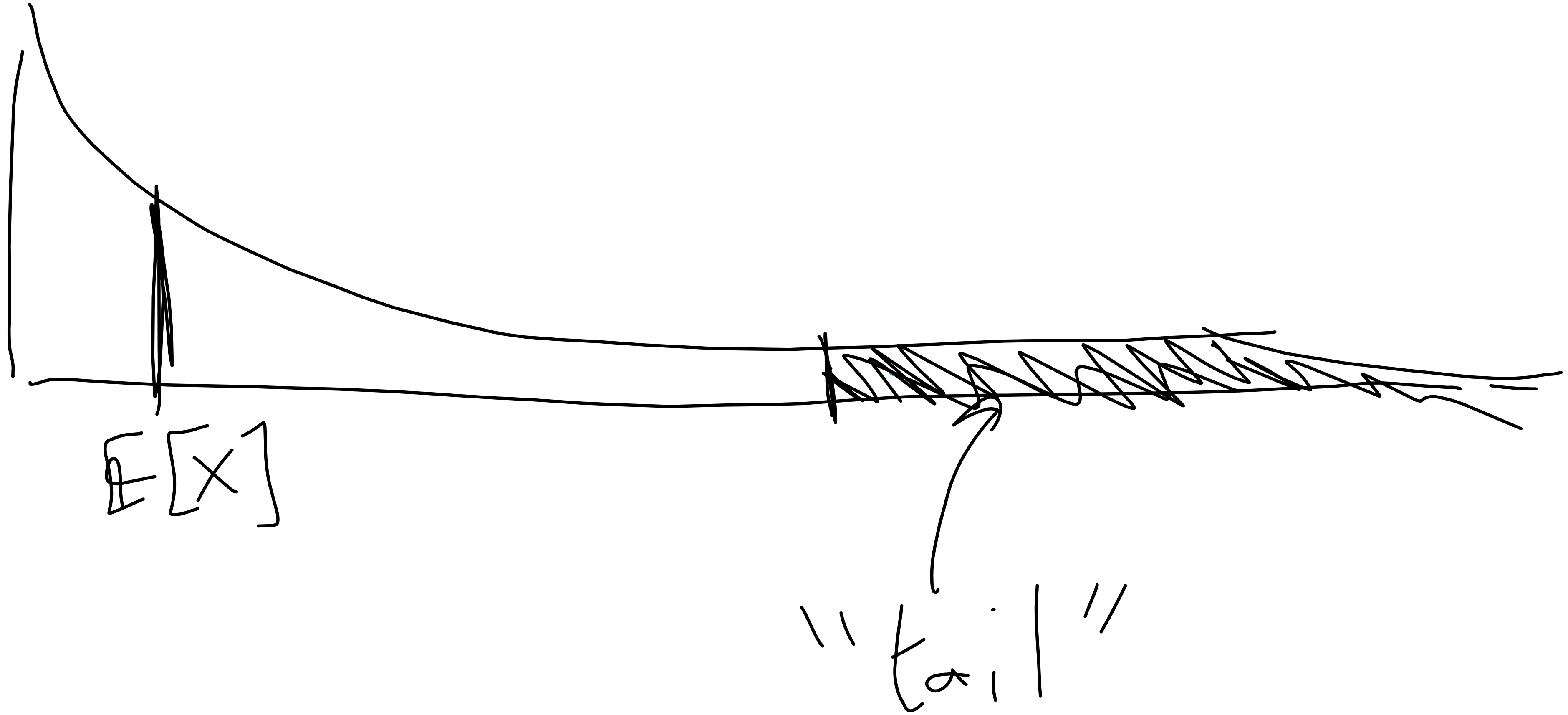
Let X be a random variable supported (only) on non-negative numbers. For any $k > 0$

$$\mathbb{P}(X \geq k\mathbb{E}[X]) \leq \frac{1}{k}$$

To apply this bound you only need to know:

1. it's non-negative
2. Its expectation.

One-sided Tail



Proof

$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot \mathbb{P}(X = x)$

$$= \sum_{x:x \geq t} x \cdot \mathbb{P}(X = x) + \sum_{x:x<t} x \cdot \mathbb{P}(X = x)$$

$$\geq \sum_{x:x \geq t} x \cdot \mathbb{P}(X = x) + 0$$

$x \geq 0$ whenever $\mathbb{P}(X = x) > 0$

$$\geq \sum_{x:x \geq t} t \cdot \mathbb{P}(X = x)$$

$$= t \cdot \sum_{x:x \geq t} \mathbb{P}(X = x)$$

$$= t \cdot \mathbb{P}(X \geq t)$$

$\mathbb{E}[X] \geq t \cdot \mathbb{P}(X \geq t)$

Markov's Inequality

Let X be a random variable supported (only) on non-negative numbers. For any $t > 0$

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

Example with geometric RV

Suppose you roll a fair (6-sided) die until you see a 6. Let X be the number of rolls.

Bound the probability that $X \geq 12$

$$\mathbb{P}(X \geq 12) \leq \frac{\mathbb{E}[X]}{12} = \frac{6}{12} = \frac{1}{2}.$$

Exact probability?

$$1 - \mathbb{P}(X < 12) \approx 1 - 0.865 = .135$$

Markov's Inequality

Let X be a random variable supported (only) on non-negative numbers. For any $t > 0$

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

A Second Example

Suppose the average number of ads you see on a website is 25. Give an upper bound on the probability of seeing a website with 75 or more ads.

Markov's Inequality

Let X be a random variable supported (only) on non-negative numbers. For any $t > 0$

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

A Second Example (answer)

Suppose the average number of ads you see on a website is 25. Give an upper bound on the probability of seeing a website with 75 or more ads.

$$\mathbb{P}(X \geq 75) \leq \frac{\mathbb{E}[X]}{75} = \frac{25}{75} = \frac{1}{3}$$

Markov's Inequality

Let X be a random variable supported (only) on non-negative numbers. For any $t > 0$

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

Useless Example

Suppose the average number of ads you see on a website is 25. Give an upper bound on the probability of seeing a website with 20 or more ads.

Markov's Inequality

Let X be a random variable supported (only) on non-negative numbers. For any $t > 0$

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

Useless Example (answer)

Suppose the average number of ads you see on a website is 25. Give an upper bound on the probability of seeing a website with 20 or more ads.

$$\mathbb{P}(X \geq 20) \leq \frac{\mathbb{E}[X]}{20} = \frac{25}{20} = 1.25$$

Well, that's…true. Technically.

But without more information we couldn't hope to do much better. What if every page gives exactly 25 ads? Then the probability really is 1.

So…what do we do?

A better inequality!

We're trying to bound the tails of the distribution.

What parameter of a random variable describes the tails?

The variance!

Chebyshev's Inequality

Two statements are equivalent.
Left form is often easier to use.
Right form is more intuitive.

Chebyshev's Inequality

Let X be a random variable. For any $t > 0$

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}$$

Chebyshev's Inequality

Let X be a random variable. For any $\mathbf{k} > 0$

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq k\sqrt{\mathrm{Var}(X)}\right) \leq \frac{1}{k^2}$$

Proof of Chebyshev

Chebyshev's Inequality

Let X be a random variable. For any $t > 0$

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}(X)}{t^2}$$

Let $Z = X - \mathbb{E}[X]$

Markov's Inequality

$\mathbb{E}[Z] = 0$

$$\mathbb{P}(|Z| \geq t) = \mathbb{P}(Z^2 \geq t^2) \leq \frac{\mathbb{E}[Z^2]}{t^2} = \frac{\mathbb{E}[Z^2] - (\mathbb{E}[Z])^2}{t^2} = \frac{\text{Var}(Z)}{t^2} = \frac{\text{Var}(X)}{t^2}$$

Inequalities are equivalent (square each side).

Z is just X shifted. Variance is unchanged.

Example with geometric RV (Chebyshev)

Suppose you roll a fair (6-sided) die until you see a 6. Let X be the number of rolls.

Bound the probability that $X \geq 12$

$$\mathbb{P}(X \geq 12) \leq \mathbb{P}(|X - 6| \geq 6) \leq \frac{\frac{5/6}{1/36}}{6^2} = \frac{5}{6}$$

Not any better than Markov ☹

Chebyshev's Inequality

Let X be a random variable. For any $t > 0$

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}(X)}{t^2}$$

Example with geometric RV (general p)

Let X be a geometric rv with parameter p

Bound the probability that $X \geq \frac{2}{p}$

$$\mathbb{P}(X \geq 2/p) \leq \mathbb{P}(|X - 1/p| \geq 1/p) \leq \frac{\frac{1-p}{p^2}}{1/p^2} = 1 - p$$

Markov gives:

$$\mathbb{P}\left(X \geq \frac{2}{p}\right) = \frac{\mathbb{E}[X]}{2/p} = \frac{1}{p} \cdot \frac{p}{2} = \frac{1}{2}.$$

For large p, Chebyshev is better.

Chebyshev's Inequality

Let X be a random variable. For any $t > 0$

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}$$

Better Example

Suppose the average number of ads you see on a website is 25. And the variance of the number of ads is 16. Give an upper bound on the probability of seeing a website with 30 or more ads.

Better Example (answer)

Suppose the average number of ads you see on a website is 25. And the variance of the number of ads is 16. Give an upper bound on the probability of seeing a website with 30 or more ads.

$$\mathbb{P}(X \geq 30) = \mathbb{P}(X - 25 \geq 30 - 25) \leq \mathbb{P}(|X - 25| \geq 5) \leq \frac{16}{25}$$

Chebyshev's – Repeated Experiments

How many coin flips (each head with probability p) are needed until you get n heads.

Let X be the number necessary. What is probability $X \geq 2n/p$?

Markov

Chebyshev

Chebyshev's – Repeated Experiments answer

How many coin flips (each head with probability p) are needed until you get n heads.

Let X be the number necessary. What is probability $X \geq 2n/p$?

Markov $\quad \mathbb{P}\left(X \geq \frac{2n}{p}\right) \leq \frac{n/p}{2n/p} = \frac{1}{2}$

Chebyshev $\quad \mathbb{P}\left(X \geq \frac{2n}{p}\right) \leq \mathbb{P}\left(\left|X - \frac{n}{p}\right| \geq \frac{n}{p}\right) \leq \frac{\text{Var}(X)}{n^2/p^2} = \frac{n(1-p)/p^2}{n^2/p^2} = \frac{1-p}{n}$

Tail Bounds – Takeaways

Useful when an experiment is complicated and you just need the probability to be small (you don't need the exact value).

Choosing a minimum n for a poll – don't need exact probability of failure, just to make sure it's small.

Designing probabilistic algorithms – just need a guarantee that they'll be extremely accurate

Learning more about the situation (e.g. learning variance instead of just mean) usually lets you get more accurate bounds.

Next time: more assumptions to get better bounds.