No paper handouts

↳ PDF on calendar.

Exponential and Normal RVs

CSE 312 Winter 26

Lecture 16

Announcements

Midterm logistics posted later today.

We have a form to fill out to:

- Tell us about a schedule conflict
- Tell us if you prefer a left- or right-handed desk (we're going to assign seats).

Practice material also posted

Continuous Zoo

$X \sim \text{Unif}(a, b)$	$X \sim \text{Exp}(\lambda)$	$X \sim \mathcal{N}(\mu, \sigma^2)$
$f_X(k) = \frac{1}{b-a}$	$f_X(k) = \lambda e^{-\lambda k}$ for $k \geq 0$	$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
$\mathbb{E}[X] = \frac{a+b}{2}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\mathbb{E}[X] = \mu$
$\text{Var}(X) = \frac{(b-a)^2}{12}$	$\text{Var}(X) = \frac{1}{\lambda^2}$	$\text{Var}(X) = \sigma^2$

It's a smaller zoo, but it's just as much fun!

Exponential

$X \sim \text{Exp}(\lambda)$

Parameter $\lambda \geq 0$ is the average number of events in a unit of time.

$$f_X(k) = \begin{cases} \lambda e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(k) = \begin{cases} 1 - e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

A continuous memoryless RV?

Poisson random variables come from a memoryless-type process.

Number of earthquakes (people in bakery, days with snow) would be memoryless under assumption that events are independent of each other!

Same experiments, but now ask a different question:

Poisson: how many incidents occur in fixed interval?

Exponential: how long do I have to wait to see the next incident?

Exponential Random Variable

Like a geometric random variable, but continuous time. How long do we wait until an event happens? (instead of "how many flips until a heads")

Where waiting doesn't make the event happen any sooner.

Geometric: $\mathbb{P}(X = k + 1 \,|X \geq 1) = \mathbb{P}(Y = k)$

When the first flip is tails, the coin doesn't remember it came up tails, you've made no progress.

For an exponential random variable:

$\mathbb{P}(X \geq k + 1 \,|X \geq 1) = \mathbb{P}(Y \geq k)$

Exponential random variable

If you take a Poisson random variable and ask “what’s the time until the next event” you get an exponential distribution!

Let’s find the CDF for an exponential.

Let $Y \sim \mathrm{Exp}(\lambda)$, be the time until the first event, when we see an average of λ events per time unit.

What’s $\mathbb{P}(Y > t)$?

What Poisson are we waiting on, and what event for it tells you that $Y > t$?

Exponential random variable

If you take a Poisson random variable and ask “what’s the time until the next event” you get an exponential distribution!

Let’s find the CDF for an exponential.

Let $Y \sim \mathrm{Exp}(\lambda)$, be the time until the first event, when we see an average of λ events per time unit. What’s $\mathbb{P}(Y > t)$?

What Poisson are we waiting on? For $X \sim \mathrm{Poi}(\lambda t)$ $\mathbb{P}(Y > t) = \mathbb{P}(X = 0)$

$$\mathbb{P}(X = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

$F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$ (for $t \geq 0$, $F_Y(x) = 0$ for $x < 0$)

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$

What's the density?

$f_Y(t) =$

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$

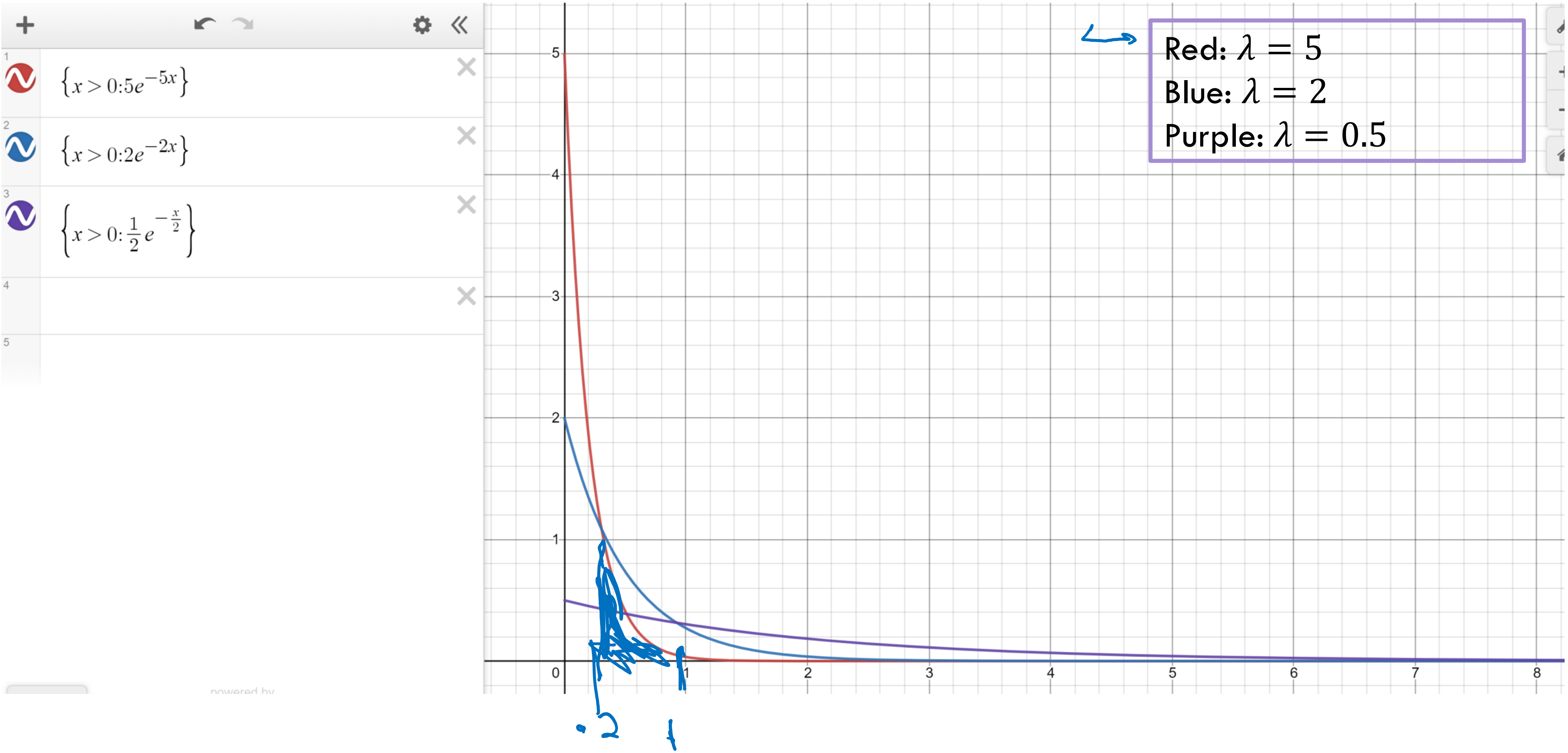
What's the density?

$$f_Y(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right) = 0 - \frac{d}{dt}\left(e^{-\lambda t}\right) = \lambda e^{-\lambda t}.$$

For t ≥ 0 it's that expression

For $t < 0$ it's just 0.

Exponential PDF $\lambda e^{-\lambda t}$



Memorylessness

$\mathbb{P}(X \geq k+1)$

$$\mathbb{P}(X \geq k+1 | X \geq 1) = \frac{\mathbb{P}(X \geq k+1 \cap X \geq 1)}{\mathbb{P}(X \geq 1)} = \frac{\mathbb{P}(X \geq k+1)}{1-(1-e^{-\lambda \cdot 1})}$$

$$= \frac{e^{-\lambda(k+1)}}{e^{-\lambda}} = e^{-\lambda k}$$

$F_X(j) = P(X \leq j)$

$1 - F_X(j) = \mathbb{P}(X > j)$

What about $\mathbb{P}(X \geq k)$ (without conditioning on the first step)?

$1-(1-e^{-\lambda k}) = e^{-\lambda k}$

It's the same!!!

More generally, for an exponential rv X, $\mathbb{P}(X \geq s+t | X \geq s) = \mathbb{P}(X \geq t)$

Side note

$1 - F_X(1)$

I hid a trick in that algebra,

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X \leq 1)$$

The first step is the complementary law.

The second step is using that $\int_1^1 f_X(z)\mathrm{d}z = 0$

In general, for continuous random variables we can switch out $\leq$ and $<$ without anything changing.

We can't make those switches for discrete random variables.

Expectation of an exponential

Don't worry about the derivation (it's here if you're interested; you're not responsible for the derivation. Just the value.

Let $X \sim \text{Exp}(\lambda)$

$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z) \text{dz}$

$= \int_0^{\infty} z \cdot \lambda e^{-\lambda z}\, dz$

Let $u = z;\ dv = \lambda e^{-\lambda z} dz\ (v = -e^{-\lambda z})$

Integrate by parts: $-ze^{-\lambda z} - \int -e^{-\lambda z}\, dz = -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z}$

Definite Integral: $-ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z} \Big|_{z=0}^{\infty} = (\lim_{z \to \infty} -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z}) - (0 - \frac{1}{\lambda})$

By L'Hopital's Rule $(\lim_{z \to \infty} -\frac{z}{e^{\lambda z}} - \frac{1}{\lambda e^{\lambda z}}) - (0 - \frac{1}{\lambda}) = \left(\lim_{z \to \infty} -\frac{1}{\lambda e^{\lambda z}}\right) + \frac{1}{\lambda} = \frac{1}{\lambda}$

Variance of an exponential

If $X \sim \text{Exp}(\lambda)$ then $\text{Var}(X) = \frac{1}{\lambda^2}$

Similar calculus tricks will get you there.

Exponential

$X \sim \text{Exp}(\lambda)$

Parameter $\lambda \geq 0$ is the average number of events in a unit of time.

$$f_X(k) = \begin{cases} \lambda e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(k) = \begin{cases} 1 - e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

The Normal (Gaussian) Distribution

The Normal distribution

Our next random variable is weird in a lot of ways:

Complicated PDF

Yet more complicated CDF

It is extremely important

"real world" data under many circumstances looks like a Gaussian distribution (a "bell cure") is one of these distributions.

Normal Random Variable

$Var(X) = \sigma^2$

$\sqrt{Var(X)} = \sigma$ "standard dev."

X is a normal (aka Gaussian) random variable with mean μ and variance σ^2 (written $X \sim \mathcal{N}(\mu, \sigma^2)$) if it has the density:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Let's get some intuition for that density...

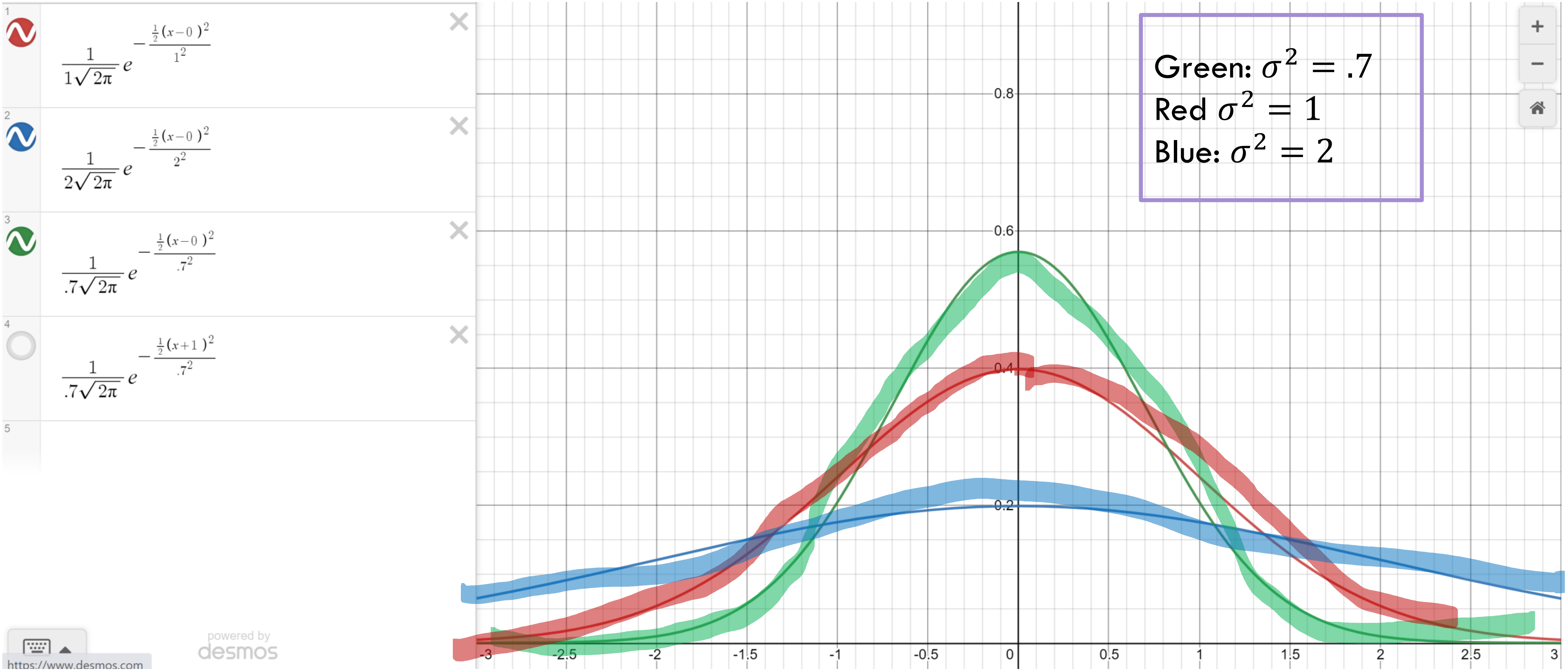
Is $\mathbb{E}[X] = \mu$?

Yes! Plug in $\mu - k$ and $\mu + k$ and you'll get the same density for every k. The density is symmetric around μ. The expectation must be μ.

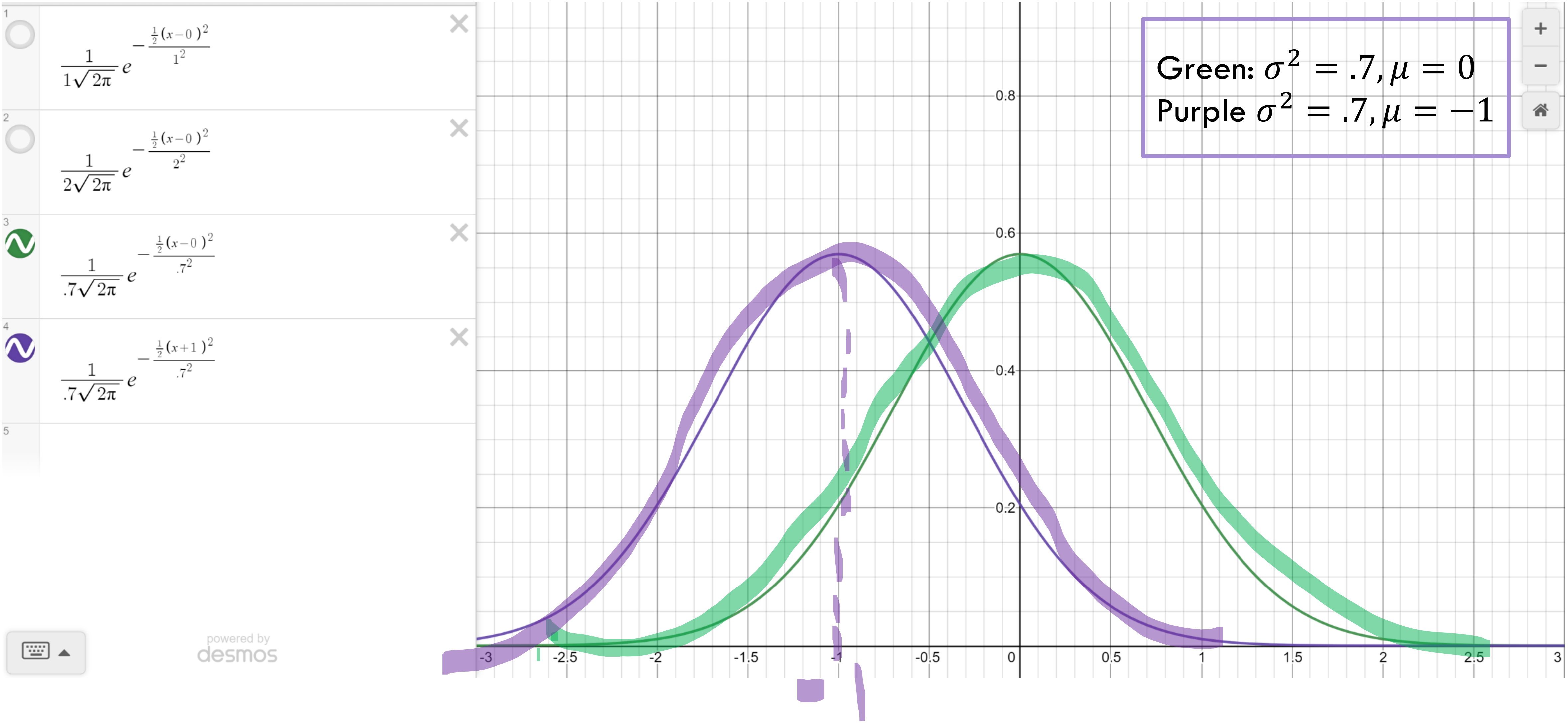
Breaking down the density

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Changing the variance



Changing the mean



Scaling Normals

When we scale a normal (multiplying by a constant or adding a constant) we get a normal random variable back!

If $X \sim \mathcal{N}(\mu, \sigma^2)$

Then for $Y = aX + b$, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Normals are special in that you get a NORMAL back.

If you multiply a binomial by $3/2$ you don't get a binomial (its support isn't even integers!)

Standardize

To turn $\mathrm{X}\sim\mathcal{N}(\mu,\sigma^2)$ into $\mathrm{Y}\sim\mathcal{N}(0,1)$ you want to set

$$Y = \frac{X-\mu}{\sigma}$$

Why standardize?

The density is a mess. The CDF does not have a pretty closed form.

But we're going to need the CDF a lot, so…

Table of Standard Normal CDF

The way we'll evaluate the CDF of a normal is to:

1. convert to a standard normal
2. Round the "z-score" to the hundredths place.
3. Look up the value in the table.

It's 2025, we're using a table?

The table makes sure we have consistent rounding rules (makes it easier for us to debug with you).
You can't evaluate this by hand – the "z-score" can give you intuition right away.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Use the table!

We'll use the notation $\Phi(z)$ to mean $F_X(z)$ where $X \sim \mathcal{N}(0,1)$.

Let $Y \sim \mathcal{N}(5,4)$ what is $\mathbb{P}(Y > 9)$?

$\mathbb{P}(Y > 9)$

$= \mathbb{P}\left(\frac{Y-5}{2} > \frac{9-5}{2}\right)$ we've just written the inequality in a weird way.

$= \mathbb{P}(X > \frac{9-5}{2})$ where X is $\mathcal{N}(0,1)$.

$= 1 - \mathbb{P}\left(X \leq \frac{9-5}{2}\right) = 1 - \Phi(2.00) = 1 - 0.97725 = .02275.$

Normal (aka Guassian)

$X \sim \mathcal{N}(\mu, \sigma^2)$

Parameter μ is the expectation; σ^2 is the variance.

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$F_X(k)$ has no nice closed form. Use the table.

$\mathbb{E}[X] = \mu$

$\mathrm{Var}(X) = \sigma^2$

More practice

Let $X \sim \mathcal{N}(3, 2)$.

What is the probability that $1 \leq X \leq 4$

More practice

Let $X \sim \mathcal{N}(3, 2)$.

What is the probability that $1 \leq X \leq 4$

$$\mathbb{P}(1 \leq X \leq 4)$$

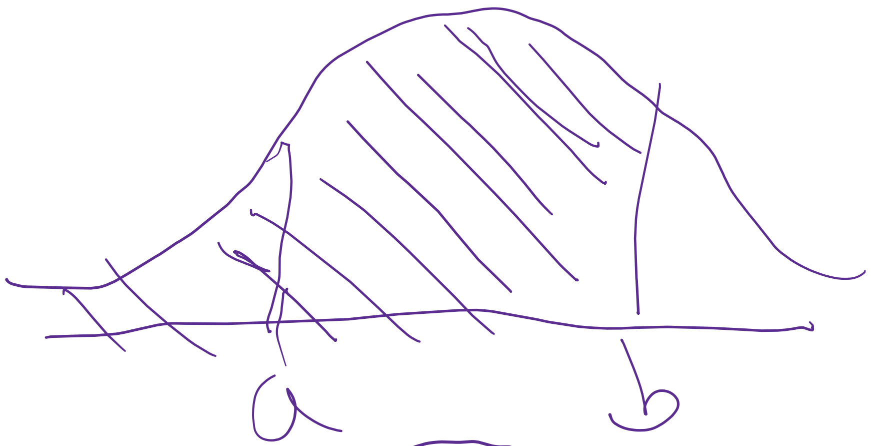
$$= \mathbb{P}\left(\frac{1-3}{\sqrt{2}} \leq \frac{X-3}{\sqrt{2}} \leq \frac{4-3}{\sqrt{2}}\right)$$

$$\approx \mathbb{P}\left(-1.41 \leq \frac{X-3}{\sqrt{2}} \leq .71\right)$$

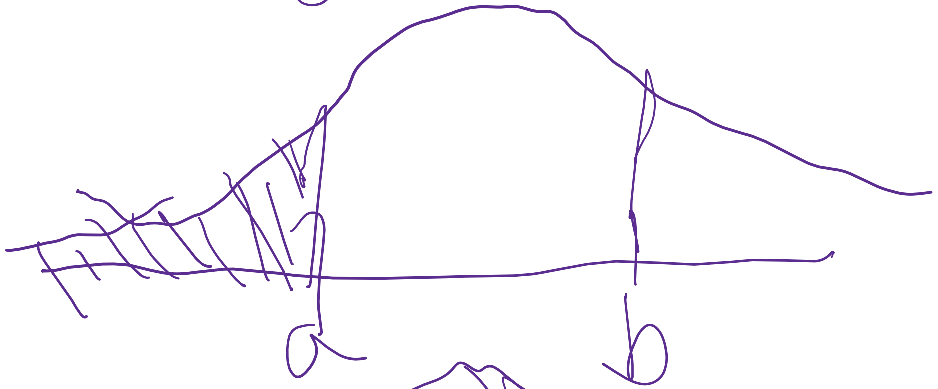
$$= \Phi(.71) - \Phi(-1.41)$$

$$= \Phi(.71) - \big(1 - \Phi(1.41)\big) = .76115 - (1 - .92073) = .68188.$$

$P(a \le X \le b)$



$F_X(b)$ $\quad P(X \le b)$



$F_X(a)$ $\quad P(X \le a)$

$F_X(b) - F_X(a)$ $\quad P(a \le X \le b)$

In real life

What’s the probability of being at most two standard deviations from the mean?

$$= \Phi(2) - \Phi(-2)$$

$$= \Phi(2) - \left(1 - \Phi(2)\right)$$

$$= .97725 - (1 - .97725) = .9545$$

You’ll sometimes hear statisticians refer to the “68-95-99.7 rule” which is the probability of being within 1,2, or 3 standard deviations of the mean.

One last (unrelated) thing

A Side Note

Make sure you understand the difference between scaling a random variable and adding up iid copies of a random variable.

If X is the result of rolling a die

$X + X$ or equivalently $2X$ says "take the result of the (one) die roll and double it"

$2X$ has support $\{2,4,6,8,10,12\}$, there's no way to get 7 because you just double the one die roll.

If X_1, X_2 are <u>independent</u> dice rolls (i.i.d.) then

$X_1 + X_2$ says "roll two dice and add their results"

$\mathbb{E}[X_1 + X_2] = \mathbb{E}[2X]$ but $\text{Var}(\text{X}_1 + \text{X}_2) \neq \text{Var}(2X)$