Expectation of a function

For any function g and any continuous random variable, X:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(X(z)) \cdot f_X(z)\,\mathrm{d}z$$

Again, analogous to the discrete case; just replace summation with integration and pmf with the pdf.

We're going to treat this as a definition.

Technically, this is really a theorem; since $f()$ is the pdf of X and it only gives relative likelihoods for X, we need a proof to guarantee it "works" for $g(X)$.

Sometimes called "Law of the Unconscious Statistician."

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What about $\mathbb{E}[g(X)]$

Let $X \sim \mathrm{Unif}(a, b)$, what about $\mathbb{E}[X^2]$?

$\mathbb{E}[X^2] =$

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Continuous Zoo

$X\sim\text{Unif}(a,b)$	$X\sim\text{Exp}(\lambda)$	$X\sim\mathcal{N}(\mu,\sigma^2)$
$f_X(k)=\frac{1}{b-a}$ $\mathbb{E}[X]=\frac{a+b}{2}$ $\text{Var}(X)=\frac{(b-a)^2}{12}$	$f_X(k)=\lambda e^{-\lambda k}$ for $k\geq 0$ $\mathbb{E}[X]=\frac{1}{\lambda}$ $\text{Var}(X)=\frac{1}{\lambda^2}$	$f_X(k)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $\mathbb{E}[X]=\mu$ $\text{Var}(X)=\sigma^2$

It's a smaller zoo, but it's just as much fun!

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Comparing Discrete and Continuous

	Discrete Random Variables	Continuous Random Variables
Probability 0	Equivalent to impossible	All impossible events have probability 0, but not conversely.
Relative Chances	PMF: $p_X(k)=\mathbb{P}(X=k)$	PDF $f_X(k)$ gives chances relative to $f_X(k')$
Events	Sum over PMF to get probability	Integrate PDF to get probability
Convert from CDF to P(M/D)F	Sum up PMF to get CDF. Look for "breakpoints" in CDF to get PMF.	Integrate PDF to get CDF. Differentiate CDF to get PDF.
$\mathbb{E}[X]$	$\sum_\omega X(\omega)\cdot p_X(\omega)$	$\int_{-\infty}^{\infty} z\cdot f_X(z)\,\mathrm{d}z$
$\mathbb{E}[g(X)]$	$\sum_\omega g\big(X(\omega)\big)\cdot p_X(\omega)$	$\int_{-\infty}^{\infty} g(z)\cdot f_X(z)\,\mathrm{d}z$
$\text{Var}(X)$	$\mathbb{E}[X^2]-(\mathbb{E}[X])^2$	$\mathbb{E}[X^2]-(\mathbb{E}[X])^2=\int_{-\infty}^{\infty}(z-\mathbb{E}[X])^2 f_X(z)\,\mathrm{d}z$

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