Linearity of Expectation

CSE 312 Winter 26
Lecture 11

Outline

- Linearity of expectation
  - Statement
  - Proof
  - A whole bunch of examples

Expectation

Expectation

The "expectation" (or "expected value") of a random variable X is:

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\boldsymbol{k} \in \Omega_X} \boldsymbol{k} \cdot \mathbb{P}(\boldsymbol{X} = \boldsymbol{k})$$

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\omega)$$

Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \ldots, X_n$

$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

This can be proven by induction.

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Constants are also fine:

For real numbers a, b, c

$$\mathbb{E}[aX+bY+c]=\mathbb{E}[aX]+\mathbb{E}[bY+c]$$
$$=a\mathbb{E}[X]+b\mathbb{E}[Y]+c$$

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Proof:

$$\mathbb{E}[X+Y] = \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)\big(X(\omega)+Y(\omega)\big)$$
$$= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega) + \mathbb{P}(\omega)Y(\omega)$$
$$= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega) + \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)Y(\omega)$$
$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

Let Z be the r.v. representing the total number of fish you both catch

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

Let Z be the r.v. representing the total number of fish you both catch

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

Let Z be the r.v. representing the total number of fish you both catch

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

Let Y be the r.v. representing the total number of heads

$$p_Y(y) = \begin{cases} \frac{1}{4} & \text{if } y = 0 \\ \frac{1}{2} & \text{if } y = 1 \\ \frac{1}{4} & \text{if } y = 2 \\ 0 & otherwise \end{cases}$$

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

Let Y be the r.v. representing the total number of heads

$$p_Y(y) = \begin{cases} \frac{1}{4} & \text{if } y = 0 \\ \frac{1}{2} & \text{if } y = 1 \\ \frac{1}{4} & \text{if } y = 2 \\ 0 & otherwise \end{cases}$$

$$\mathbb{E}[Y] = \Sigma_{k \in \Omega_Y} p_Y(k) \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

Make a prediction --- what should $\mathbb{E}[X]$ be?

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

$$\mathbb{E}[\boldsymbol{X}] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) =$$

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Ok, but what actually is it?
I don't have intuition for this formula.

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{E}[\boldsymbol{X}] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

$$= np(p + (1-p))^{n-1} = np$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{E}[\boldsymbol{X}] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum^{n} \ldots p)^{n-k}$$

$$= \sum_{k}^{n} \ldots$$

$$\ldots n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

$$= np(p + (1-p))^{n-1} = np$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

No one wants to do proofs like this every time!

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}]=\mathbb{E}[\boldsymbol{X}]+\mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1+X_2+\cdots+X_n]=\mathbb{E}[X_1]+\mathbb{E}[X_2]+\cdots+\mathbb{E}[X_n]$$

Indicator Random Variables

For any event A, we can define the indicator random variable $\mathbf{1}[A]$ for A

$$\mathbf{1}[A] = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(A)$$
$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(A)$$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x = 1 \\ 1 - \mathbb{P}(A) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X]$$
$$= 1 \cdot p_X(1) + 0 \cdot p_X(0)$$
$$= p_X(1) = \mathbb{P}(A)$$

You'll also see notation like:

$\mathbb{1}[A]$, 1_A, $\mathbb{1}[\text{some boolean}]$

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

What indicators can we define? What 'Booleans' have enough information to combine (add) and solve the problem?

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the ith coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X_i = 1) = p$$
$$\mathbb{P}(X_i = 0) = 1 - p$$

$$X = \sum_{i=1}^{n} X_i$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

H T H T

1 0 1 0

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

$$X = \sum_{i=1}^{n} X_i$$
$$\mathbb{E}[X_i] = p$$

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\textstyle\sum_{i=1}^{n} X_i] \\ &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ &= \textstyle\sum_{i=1}^{n} \mathbb{E}[X_i] \\ &= \textstyle\sum_{i=1}^{n} p = np\end{aligned}$$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. **Decompose:** Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + X_2 + \cdots + X_n$$

2. **LOE:** Apply Linearity of Expectation

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

3. **Conquer:** Compute the expectation of each X_i

Often X_i are indicator random variables

Pairs with the same birthday

↳ What can I tally / list?

$\frac{1}{365}$

In a class of m students, on average how many pairs of people have the same birthday?

<u>Decompose:</u>

<u>LOE:</u>

<u>Conquer:</u>

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person i, j have the same bithday} \\ 0 & \text{otherwise} \end{cases}$$

$$X = \Sigma_{i,j} X_{ij}$$

$i < j$

LOE:

Conquer:

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

<u>Decompose:</u> Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person i, j have the same bithday} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i,j} X_{ij}$$

<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}\left[\Sigma_{i,j} X_{ij}\right] = \Sigma_{i,j} \mathbb{E}\left[X_{ij}\right]$$

<u>Conquer:</u>

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

<u>Decompose:</u> Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person i, j have the same bithday} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i,j} X_{ij}$$

<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}\left[\Sigma_{i,j} X_{ij}\right] = \Sigma_{i,j} \mathbb{E}\left[X_{ij}\right]$$

<u>Conquer:</u>

$$\mathbb{E}\left[X_{ij}\right] = \mathbb{P}\left(X_{ij} = 1\right) = \frac{365}{365 \cdot 365} = \frac{1}{365}$$

$$\mathbb{E}[X] = \binom{m}{2} \cdot \mathbb{E}\left[X_{ij}\right] = \binom{m}{2} \cdot \frac{1}{365}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

Let X be the number of people that end up in front of their own name tag. **Find $\mathbb{E}[X]$.**

<u>Decompose:</u>

What X_i can we define that have the needed information?

<u>LOE:</u>

<u>Conquer:</u>

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

X is the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

<u>Decompose:</u> Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person i sits infront of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = \Sigma_{i=1}^{n} X_i$

<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

<u>Conquer:</u>

These X_i are not independent!
That's ok!!

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

X is the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

<u>Decompose:</u> Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person i sits infront of their own name tag} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i=1}^{n} X_i$$

<u>LOE:</u>

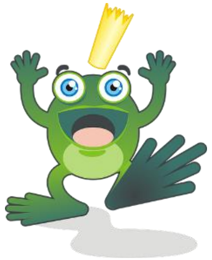
$$\mathbb{E}[X] = \mathbb{E}[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

<u>Conquer:</u>

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \qquad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$

Extra Practice

Frogger

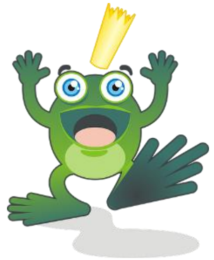


A frog starts on a 1-dimensional number line at 0.

Each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$.

After 2 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

Frogger – Brute Force



A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

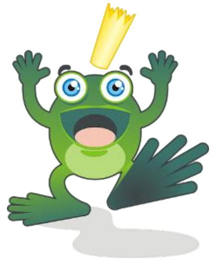
We could find the PMF by computing the probability for each value in the range of X, and then applying definition of expectation:

$$p_X(x) = \begin{cases} p_L^2 & x = -2 \\ 2p_L p_S & x = -1 \\ 2p_L p_R + p_S^2 & x = 0 \\ 2p_R p_S & x = 1 \\ p_R^2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

We think about the outcomes that correspond to each value of X and compute the probability of that. For example, X=0 happens when the frog is at the same position after 2 sec – this means it either moved left and then right, or right and then left, or did not move both seconds.

$$\mathbb{E}[\boldsymbol{X}] = \Sigma_\omega P(\omega)X(\omega) = (-2)p_L^2 + (-1)2p_L p_S + 0 \cdot (2p_L p_R + p_S^2) + (1)2p_R p_S + (2)p_R^2 = 2(p_R - p_L)$$

Frogger – LOE



Or we can apply LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

Define X_i as follows:

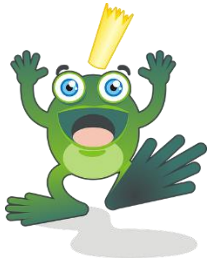
$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{2} X_i\right] = \sum_{i=1}^{2} \mathbb{E}[X_i] = 2(p_R - p_L)$$

Frogger – LOE



If we interested in a whole minute (60 sec), the first approach would be awful because we would need to compute many probabilities or deal with a gnarly summation! Instead, we can use LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 60 seconds, let X be the location of the frog. Find $\mathbb{E}[X]$.

Define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{60} X_i\right] = \sum_{i=1}^{60} \mathbb{E}[X_i] = \mathbf{60}(p_R - p_L)$$