Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

Note: X **and** Y **do not have to be independent**

Proof:

$$\begin{aligned}\mathbb{E}[X+Y] &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)\big(X(\omega)+Y(\omega)\big)\\ &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega)+\mathbb{P}(\omega)Y(\omega)\\ &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega)+\Sigma_{\omega\in\Omega}\mathbb{P}(\omega)Y(\omega)\\ &= \mathbb{E}[X]+\mathbb{E}[Y]\end{aligned}$$

Indicator Random Variables

For any event A, we can define the indicator random variable $\mathbf{1}[A]$ for A

$$\mathbf{1}[A] = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X=1) = \mathbb{P}(A)$$
$$\mathbb{P}(X=0) = 1-\mathbb{P}(A)$$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x=1 \\ 1-\mathbb{P}(A) & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}&\mathbb{E}\,[X]\\ &= 1\cdot p_X(1) + 0\cdot p_X(0)\\ &= p_X(1) = \mathbb{P}(A)\end{aligned}$$

You'll also see notation like:

$\mathbb{1}[A]$, 1_A, $\mathbb{1}[\text{some boolean}]$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. Decompose: Finding the right way to decompose the random variable into sum of simple random variables
$$X = X_1 + X_2 + \cdots + X_n$$
2. LOE: Apply Linearity of Expectation
$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$
3. Conquer: Compute the expectation of each X_i

Often X_i are indicator random variables

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Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

LOE:

Conquer:

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