Probability Definitions

CSE 312 Winter 26
Lecture 5

Extra Practice

Practice

How do we know which rule to apply?

With practice you can pick out patterns for which ones might be plausible.

But if as you’re working you realize things are getting out of control, put it aside and try something different.

Cards

A “standard” deck of cards has 52 cards. Each card has a suit

diamonds ♦,

hearts ♥,

clubs ♧,

spades ♠

and a value (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

A “5-card-hand” is a set of 5 cards

How many five-card “flushes” are there? – a flush is a hand of cards all of the same suit.

Cards

How many five-card “flushes” are there? – a flush is a hand of cards all of the same suit.

Way 1: How can I describe a flush? Which suit it is, and which values:

$$\binom{4}{1} \cdot \binom{13}{5}$$

Cards

Way 2: Pretend order matters. The first card can be anything,

After that, you'll have 12 options (the remaining cards of the suit), then 11, …

Then divide by $5!$, since order isn't supposed to matter.

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

This is the same number as what we got on the last slide!

A Solution with a Problem

You wish to count the number of 5-card hands with at least 3 aces.

There are 4 Aces (and 48 non aces)

$\binom{4}{3} \cdot \binom{49}{2}$

Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

What's wrong with this calculation?

What's the right answer?

A Solution with a Problem

For a hand, there should be exactly one set of choices in the sequential process that gets us there.

{A♧, A♠, A♦} {A♥, K ♠}

And

{A♧, A♠, A♥}, {A♦, K ♠}

Are two different choices of the process, but they lead to the same hand!

A Solution with a Problem

We could count exactly which hands appear more than once, and how many times each appears and compensate for it.

See the extra slides at the end.

An easier solution is to try again…

The problem was trying to account for the “at least” – come up with disjoint sets and count separately.

$$\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$$

If there are exactly 3 aces, we choose which 3 of the 4, then choose which 2 cards among the 48 non-aces. If all 4 aces appear, then one of the remaining 48 cards finishes the hand. Applying the sum rule completes the calculation.

Takeaway

It's hard to count sets where one of the conditions is "at least X"

You usually need to break those conditions up into disjoint sets and use the sum rule.

Probability

Probability

Probability is a way of **quantifying** under uncertainty.

When more than one outcome is possible,

To have "real-world" examples, we'll need to start with some foundational processes that we're going to assert exist

- We can flip a coin, and each face is equally likely to come up
- We can roll a die, and every number is equally likely to come up
- We can shuffle a deck of cards so that every ordering is equally likely.

Sample Space

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

"outcome" is our word for a single element of Ω.

Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \{HH, HT, TH, TT\}$

For rolling a (normal) die: $\Omega = \{1,2,3,4,5,6\}$

Events

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

Examples:

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

Examples

I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.

The event "the sum of the dice is even" is in gold

The event "the blue die is 1" is in green

	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Probability

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome is.

We'll define a function

$\mathbb{P}: \Omega \to [0,1]$

i.e. $\mathbb{P}$ takes an element of Ω as input and outputs the probability of the outcome.

We'll also use $\Pr[\omega]$, $P(\omega)$ as notation.

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

It depends on what we want to model

If the coin is fair $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a biased coin: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

Probability Space

Probability Space

A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

Ω is the sample space

$\mathbb{P}: \Omega \to [0, 1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$
- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Probability Space

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is this a valid probability space?

$\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$.

Measure

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

So what's the probability of seeing a heads?

Seeing heads isn't an element of the sample space!

Define $\mathbb{P}(E) = \sum_{\omega \in \mathrm{E}} \mathbb{P}(\omega)$

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Uniform Probability Space

The most common probability measure is the **uniform** probability measure. In the uniform measure, for every event E

$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

A. $\binom{100}{50}/2^{100}$

B. $1/101$

C. $1/2$

D. $1/2^{50}$

E. There is not enough information in this problem.

Mutually Exclusive Events

Two events E, F are mutually exclusive if they cannot happen simultaneously.

In notation, $E \cap F = \emptyset$ (i.e. they're disjoint subsets of the sample space).

For example, if $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$E_1 =$"the coin came up heads"

$E_2 =$"the coin came up tails"

$E_3 =$"the die showed an even number"

E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

Axioms and Consequences

We wrote down 3 requirements (axioms) on probability measures

- $\mathbb{P}(x) \geq 0$ for all x (non-negativity)
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$ (normalization)
- If E and F are mutually exclusive then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (countable additivity)

These lead quickly to these three corollaries

- $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$ (complementation)
- If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$ (monotonicity)
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ (inclusion-exclusion)

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space?

What is your probability measure $\mathbb{P}$?

What is your event?

What is the probability?

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space? $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? $\{2,4,6\} \times \{2,4,6\}$

What is the probability? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we can't tell the dice apart and always put the dice in increasing order by value.

What is your sample space?

$\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\}$

What is your probability measure $\mathbb{P}$?

$\mathbb{P}((x, y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x, x) = 1/36$

What is your event? $\{(2,2), (4,4), (6,6), (2,4), (2,6), (4,6)\}$

What is the probability? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will usually make finding the probabilities easier to calculate.

Non-uniform measures

Last Time

Sample Space

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Takeaways

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Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space

Probability Measure

Event

Probability

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\{(x, y) \colon x \text{ and } y \text{ are different cards } \}$

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability: $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50 * 49 * 48 * \cdots . * 2 * 1}{52 * 51 * 50 * 49 * 48 * \ldots * 2 * 1}$

Takeaway

There’s often information you “don’t need” in your sample space.

It won’t give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

Uniform Probability Space

The most common probability measure is the **uniform** probability measure. In the uniform measure, for every event E

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A. $\binom{100}{50}/2^{100}$

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Few notes about events and samples spaces

- If you’re dealing with a situation where you may be able to use a uniform probability space, make sure to set up the sample space in a way that every outcome is equally likely.

- Try not overcomplicate the sample space – only include the information that you need in it.

- When you define an event, make sure it is a <u>subset</u> of the sample space!

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if ?

$\mathbb{P}(E) = 1$ if and only if ?

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if an event can't happen.

$\mathbb{P}(E) = 1$ if and only if an event is guaranteed (every outcome outside E has probability 0).

More Practice

Probable Fruit

The fruit store sells apples, bananas, and oranges.
Robbie will buy a bag of 10 fruits (order doesn't matter) to bring to lecture, uniformly at random among all possible bags that contain at least one of each fruit type.

You and your friend are first in line to take fruit, and will take an apple each if it's available---what is the probability you both get an apple?

Probable Fruit (Sample Space)

Defining the sample space (other than in English sentences) would be annoying--we'll skip it.

But we need its size…two options

Complementary counting and Inclusion-Exclusion

Let A, B, O be bags with no apples, bananas, oranges. S be set of all bags. We want $S \setminus (A \cup B \cup O)$.

$|S| = \binom{10+3-1}{3-1}$ 10 fruit, 3 types

$|A| = \binom{10+2-1}{2-1} = |B| = |O|$ 10 fruit, but only 2 types now.

(This formula actually simplifies to 11; think about why)

$|A \cap B| = \binom{10+1-1}{1-1} = |A \cap O| = |B \cap O|$ 10 fruit, but only 1 type now.

(This formula simplifies to 1; think about why)

$|A \cap B \cap O| = 0$ no way to have no types of fruit!

Combining: $\binom{10+3-1}{3-1} - 3\binom{10+2-1}{2-1} + 3 \cdot \binom{10+1-1}{1-1} = 66 - 33 + 3 = 36$

Clever Solution

Put the one fruit of each type in the bag first. We now need 7 more fruits (with no restrictions).

$\binom{7+3-1}{3-1} = 36.$

What about the event?

We need at least 2 apples, at least 1 banana, at least 1 orange.

Could do cases again, let's use just the clever approach:

6 more fruits to add: $\binom{6+3-1}{3-1} = \binom{8}{2} = 28$

Since we're in the uniform probability space, just divide

$\frac{28}{36} = \frac{7}{9}$