More Counting

CSE 312 Winter 26
Lecture 2

Where Are We?

Last time:

Sum and Product Rules

Sequential Processes

Different ways of thinking lead to different formulas

Today:

Wrap up the book example

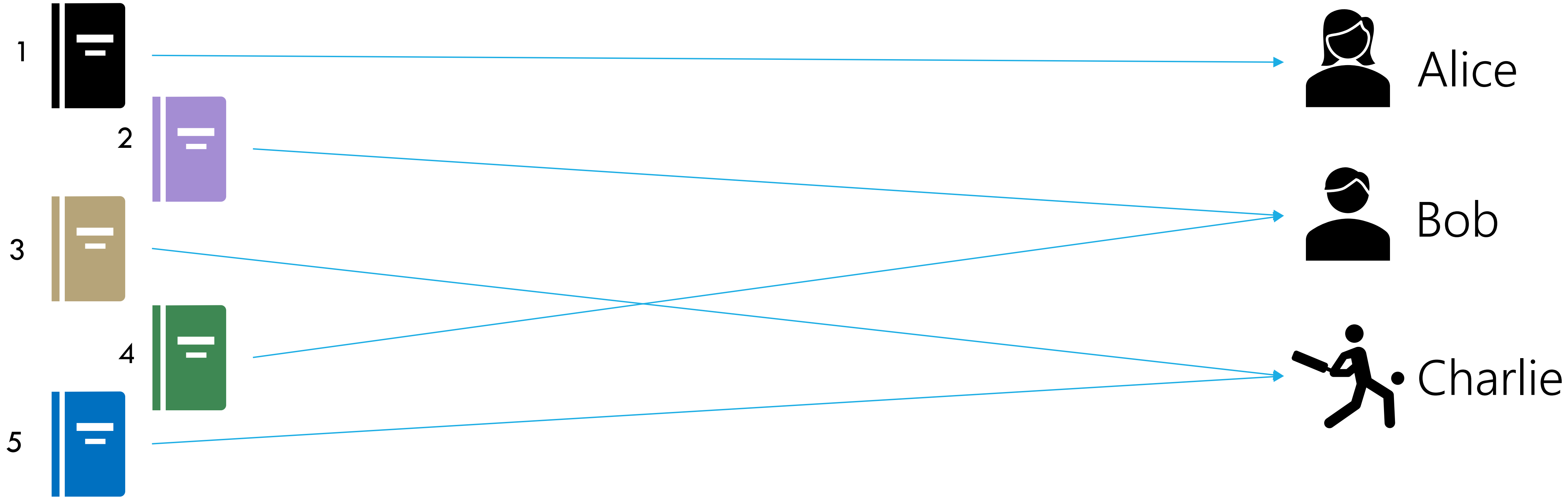
A few more rules: Combinations and Permutations.

Proof by double counting

Assigning Books

We have 5 books to split to 3 people (Alice, Bob, and Charlie)

Every book goes to exactly one person, but each person could end up with no books (or all of them, or something in between).



Assigning Books (attempt 1)

We have 5 books to split to 3 people (Alice, Bob, and Charlie)

Every book goes to exactly one person, but each person could end up with no books (or all of them, or something in between).

Attempt 1: We're choosing subsets!

Alice could get any of the $2^5 = 32$ subsets of the books.

Bob could get any of the $2^5 = 32$ subsets of the books.

Charlie could get any of the $2^5 = 32$ subsets of the books.

Total is product of those three steps $32 \cdot 32 \cdot 32 = 32768$

Analyzing Attempt 1

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We overcounted!

If Alice gets $\{1,2\}$, Bob can't get any subset, he can only get a subset of $\{3,4,5\}$. And Charlie's subset is just whatever is leftover after Alice and Bob get theirs…

Fixing All The Books (attempt 1)

You **could**

List out all the options for Alice.

For each of those (separately), list all the possible options for Bob and Charlie.

Use the Summation rule to combine.

~OR~ you could come at the problem from a different angle.

Fixing All the Books (attempt 2)

Instead of figuring out which books Alice gets, choose book by book which person they go to.

Step 1: Book 1 has 3 options (Alice, Bob, or Charlie).

Step 2: Book 2 has 3 options (A, B, or C)

…

Step 5: Book 5 has 3 options.

Total: 3^5.

More sequence practice

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$.

Pause

Questions in combinatorics and probability are often dense. A single word can totally change the answer. Does order matter or not? Are repeats allowed or not? What makes two things "count the same" or "count as different"?

Let's look for some keywords

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$.

> **Sequences** implies that order matters – $(1,2,3)$ and (2,1,3) are different.
> **Distinct** implies that you can't repeat elements (1,2,1) doesn't count.
>
> $\{1,2,3\}$ is our "universe" – our set of allowed elements.

More sequence practice (answer)

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$.

Step 1: 3 options for the first element.

Step 2: 2 (remaining) options for the second element.

Step 3: 1 (remaining) option for the third element.

$3 \cdot 2 \cdot 1$.

Factorial

That formula shows up a lot.

The number of ways to "permute" (i.e. "reorder", i.e. "list without repeats") n elements is "n factorial"

n factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$$

We only define $n!$ for natural numbers n.

As a convention, we define: 0!=1.

Distinct Letters

How many length 5 strings are there over the alphabet $\{a, b, \dots, z\}$ where each string does not repeat a letter.

E.g. "azure" is an allowed string, but "steve" is not, nor is "abcdef"

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$$

In General

k-permutation

The number of k-element sequences of distinct symbols from a universe of n symbols is:

$$P(n,k) = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

Said out loud as "P n k" or "n permute k" or "n pick k"

Alternative notation: ${}_nP_k$

Edge cases: $P(n,n) = n!$, $P(n,0) = 1$,
$P(n,k)$ for $k < 0$ or $k > n$ is undefined.

Change it slightly

How many **subsets** of size 5 are there of $\{a, b, \dots, z\}$

Remember subsets we don't count repeats – so we still have that rule.

But for subsets order doesn't matter.

$\{a, z, u, r, e\}$ is the same set as $\{a, u, r, e, z\}$ (even though "azure" and "aurez" are different strings).

Clever approach – count two ways

Let's artificially introduce a requirement that we are supposed to have an ordered list.

Then the total is going to be $P(26,5)$.

How else could we get an ordered list? With this sequential process:

Step 1: Choose a subset.

Step 2: Put the subset in order.

These better give us the same number, so:

$$\frac{26!}{(26-5)!} = ? \cdot 5!$$

Clever approach – count two ways (answer)

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Step 1: Choose a subset.

Step 2: Put the subset in order.

These better give us the same number, so:

$$\frac{26!}{(26-5)!} = ? \cdot 5!$$

So the number of size-5 subsets of a size-26 set is:

$$\frac{26!}{(26-5)!\,5!}$$

Number of Subsets

k-combination

The number of k-element subsets from a set of n symbols is:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

Said out loud "n choose k" (or sometimes: "n combination k")

Lots of notation:

${}_nC_k$ or $\binom{n}{k}$ or $C(n,k)$ all mean "number of size-k subsets of a size-n set."

Edge cases: $\binom{n}{0} = 1, \binom{n}{n} = 1$; $\binom{n}{k}$ for $k < 0$ or $k > n$ is undefined.

Second Takeaway

The second way of counting hints at a generally useful trick:

Pretend that order does matter, then divide by the number of orderings of the parts where order doesn't matter.

For example, here's another way to get the formula for combinations:

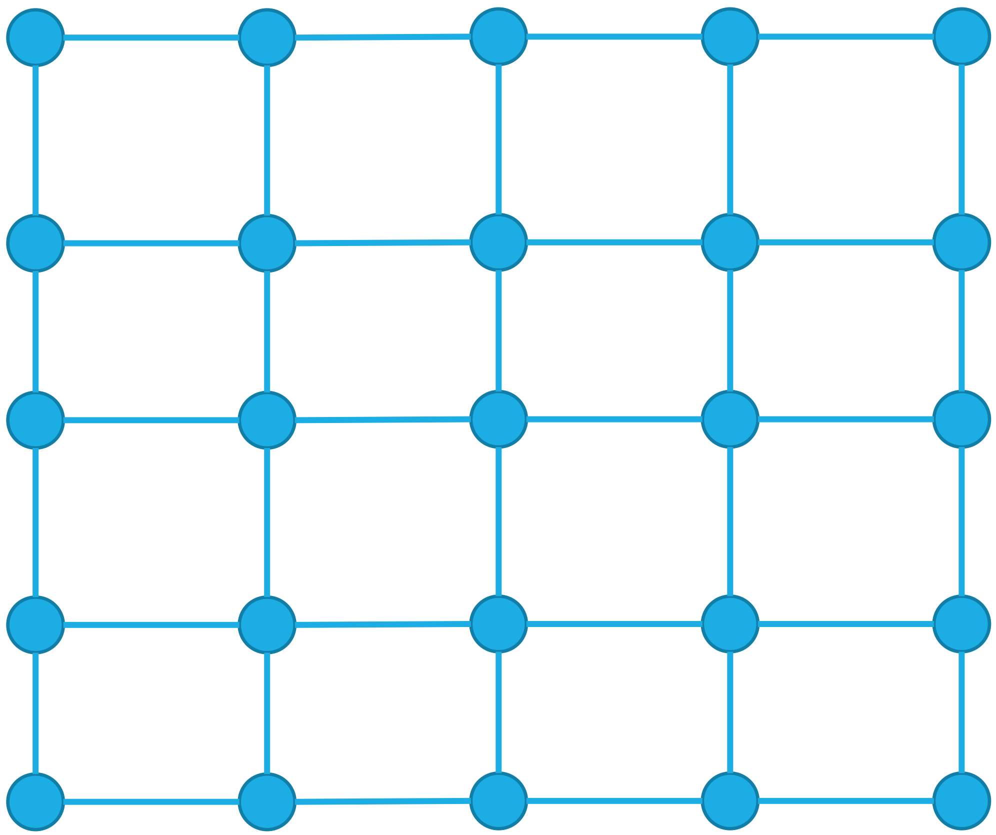
You have n elements. Put them in order, take the first k as your set.

$n!$ Orderings overall. We've overcounted because:

- Among the first k, order doesn't matter between them. Divide by $k!$.
- Among the last $n-k$, order doesn't matter between them. Divide by $(n-k)!$.

$$\frac{n!}{k!\,(n-k)!}$$

Path Counting (options)



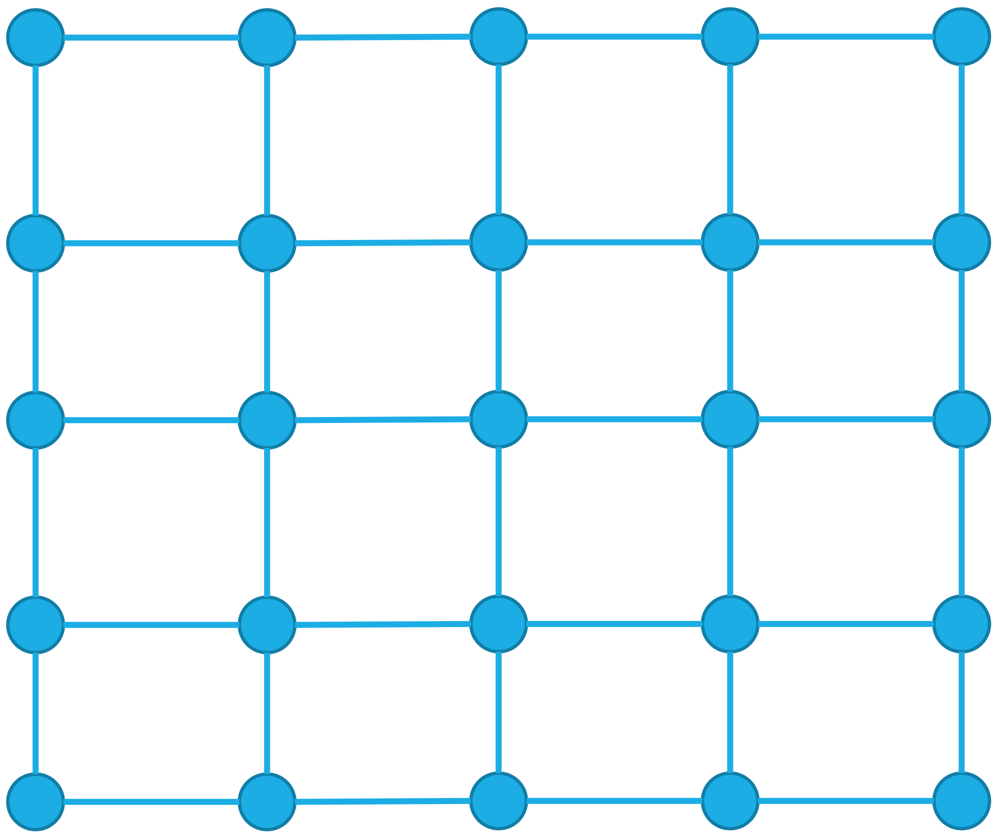
We're in the lower-left corner, and want to get to the upper-right corner.

We're only going to go right and up.

How many different paths are there?

$A.\ 2^8$
$B.\ P(8,4)$
$C.\ \binom{8}{4}$
D. Something else

Path Counting



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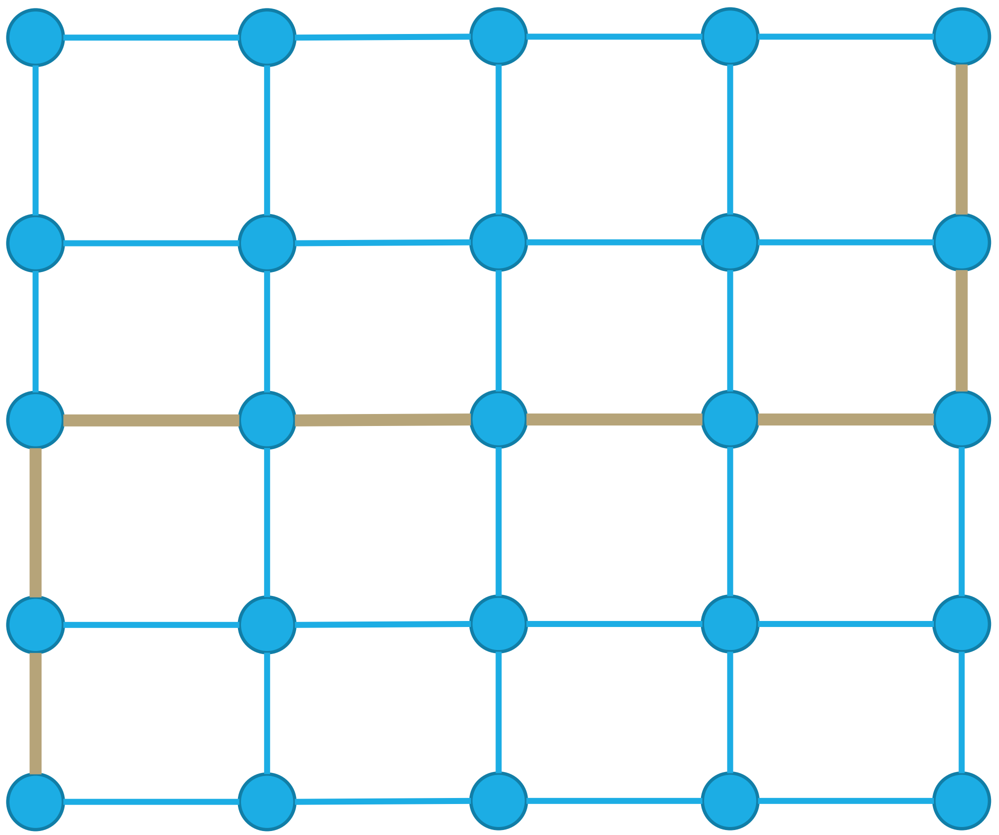
How many different paths are there?

Idea 1:
We're going to take 8 steps.
Choose which SET of 4 of the steps will be up (the others will be down).

Path Counting (Idea 1)



We're in the lower-left corner, and want to get to the upper-right corner.

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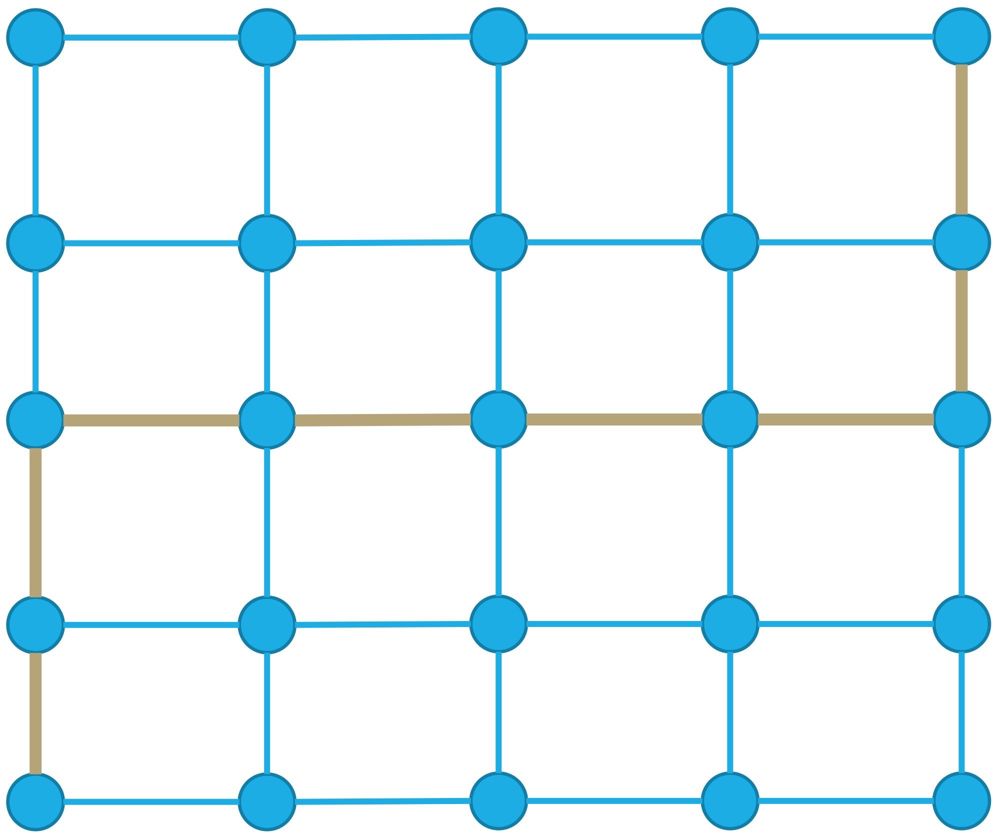
How many different paths are there?

Idea 1:
We're going to take 8 steps.
Choose which SET of 4 of the steps will be up (the others will be down).
E.g. $\{1,2,7,8\}$ is
How many size-4 subsets of $\{1,2,3,4,5,6,7,8\}$ are there?

$\binom{8}{4}$ is the answer.

Path Counting (Idea 2)



We're in the lower-left corner, and want to get to the upper-right corner.

We're only going to go right and up.

How many different paths are there?

Idea 2: Introduce artificial ordering

Order $\uparrow_A \uparrow_B \uparrow_C \uparrow_D \rightarrow_A \rightarrow_B \rightarrow_C \rightarrow_D$ $\quad 8!$

Remove the overcounting

Those $4 \uparrow$ are really the same, divide by $4!$

The $4 \rightarrow$ are really the same, divide by $4!$

Total: $\frac{8!}{4! \cdot 4!}$

$\binom{8}{4}$ is the answer.

Overcounting

How many anagrams are there of SEATTLE

(an anagram is a rearrangement of letters).

It's not 7! That counts SEATTLE and SEATTLE as different things!

I swapped the Es (or maybe the Ts)

Overcounting (answer)

How many anagrams are there of SEATTLE

Pretend the order of the Es (and Ts) relative to each other matter (that SEATTLE and SEATTLE are different)

How many arrangements of SEATTLE? $7!$

How have we overcounted? Es relative to each other and Ts relative to each other $2! \cdot 2!$

Final answer $\frac{7!}{2! \cdot 2!}$

One More Counting Technique

Complementary Counting

Count the complement of the set you're interested in.

How many length 5 strings over $\{a, b, c, \dots, z\}$ are there with **at least** 1 'a'

Let A be the set of strings we're interested in, $\mathcal{U}$ be all length 5 strings

$|A| = |\mathcal{U} \setminus \bar{A}| = |\mathcal{U}| - |\bar{A}| = 26^5 - 25^5$

Combination Facts

Some Facts about combinations

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Two Proofs of Symmetry

Proof 1: By algebra

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ Definition of Combination

$$= \frac{n!}{(n-k)!k!}$$ Algebra (commutativity of multiplication)

$$= \binom{n}{n-k}$$ Definition of Combination

Two Proofs of Symmetry (analysis)

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

But…do you know *why*? Can you *feel* why it’s true?

Combinatorial Proof of Symmetry

Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1: We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2: We choose the $n - k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

Since we're counting the same thing, the numbers must be equal.
So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ algebraic proof

$\binom{n-1}{k-1} + \binom{n-1}{k} = \dfrac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \dfrac{(n-1)!}{k!(n-1-k)!}$	definition of combination
$= \dfrac{(n-1)!}{(k-1)!(n-k)!} + \dfrac{(n-1)!}{k!(n-k-1)!}$	subtraction
$= \dfrac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!}$	Find a common denominator
$= \dfrac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!}$	factor out common terms
$= \dfrac{(n-1)!\,[k+(n-k)]}{k!(n-k)!}$	Cancel $(k-1)!(n-k-1)!$
$= \dfrac{(n-1)!\cdot n}{k!(n-k)!} = \dfrac{n!}{k!(n-k)!}$	Algebra
$= \binom{n}{k}$	Definition of combination

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ combinatorial proof

You and $n-1$ other people are trying out for a k person team. How many possible teams are there?

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Way 2: There are two types of teams. Those for which you make the team, and those for which you don't.

If you do make the team, then $k-1$ of the other $n-1$ also make it.

If you don't make the team, k of the other $n-1$ also make it.

Overall, by sum rule, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since we're computing the same number two different ways, they must be equal. So: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Takeaways

Formulas for factorial, permutations, combinations.

A useful trick for counting is to pretend order matters, then account for the overcounting at the end (by dividing out repetitions)

When trying to prove facts about counting, try to have each side of the equation count the same thing.

Much more fun and much more informative than just churning through algebra.

Extra Practice

Books, revisited

Remember the books problem from lecture 1? Books 1,2,3,4,5 need to be assigned to Alice, Bob, and Charlie (each book to exactly one person).

Now that we know combinations, try a sequential process approach. It won't be as nice as the change of perspective, but we can make it work.

Break into cases based on how many books Alice gets, use the sum rule to combine.

Books, revisited (2)

Step 1: give Alice gets 0 books (1 way to do this)

Step 2: give Bob a subset of the remaining books 2^5 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+$

Step 1: give Alice 1 book ($\binom{5}{1}$ ways to do this)

Step 2: give Bob a subset of the 4 remaining books 2^4 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+ \ldots$

Books, revisited (3)

Add all the options together

$$1 \cdot 2^5 \cdot 1 + \binom{5}{1} \cdot 2^4 \cdot 1 + \binom{5}{2} \cdot 2^3 \cdot 1 + \binom{5}{3} \cdot 2^2 \cdot 1 + \binom{5}{4} \cdot 2^1 \cdot 1 + \binom{5}{5} \cdot 2^0 \cdot 1$$

If you plug and chug, you'll get the number we got last time. It took quite a bit of work, but we got there!

Why Divide?

Why divide by 2! (not subtract)

When we had two copies of the same thing, we divided by 2!, why was it division?

It sometimes helps to see all the options. Imagine we are looking at anagrams of AABC.

If we can tell the difference between A's (one is black, the other is red), we have these 4! =24 options

Anagrams			
AABC	AABC	AACB	AACB
ABCA	ABCA	ACBA	ACBA
BCAA	BCAA	CBAA	CBAA
ABAC	ABAC	ACAB	ACAB
BAAC	BAAC	CAAB	CAAB
BCAA	BCAA	BCAA	BCAA

Why divide (in general)

If we can tell the difference between A's (one is black, the other is red), we have these 24 options

Anagrams			
AABC	~~AABC~~	AACB	~~AACB~~
ABCA	~~ABCA~~	ACBA	~~ACBA~~
BCAA	~~BCAA~~	CBAA	~~CBAA~~
ABAC	~~ABAC~~	ACAB	~~ACAB~~
BAAC	~~BAAC~~	CAAB	~~CAAB~~
BCAA	~~BCAA~~	BCAA	~~BCAA~~

But if we can't tell the difference between the A's, there's two copies of each of the strings---so there weren't 24, options but 12; if you have 2 copies of everything, dividing by 2 fixes the overcounting.

In general, with k copies of a character, you get $k!$ copies of each item (try this yourself with AAAB); dividing by $k!$ corrects the overcounting.