CSE 312 : Autumn 2025 Midterm Exam, Form A Solutions

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Instructions

- You have 80 minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper). Additionally, you will be given a formula sheet.
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- In general, show us the work you used to get to an answer; explanations will help us award partial credit, but we do not expect explanations at the level we usually require on homeworks.

Simplification Expectations

- Since you don't have a calculator for this exam, you do not have to do simplifications that could be done easily with a calculator. For example
$$\frac{\binom{5}{3} \cdot 17^2}{1-p} + 5^3$$
can be given as a final answer.
- However, answers which are much more complicated than the expected answer may receive deductions. For example: $\sum_{i=0}^{n} \binom{n}{i}$ or $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ are **not** simplified sufficiently.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	**Max points**
Terraforming Mars	12
Passbirds	12
Achoo!	17
A Duck Walks up to a Lemonade Stand	16
Multiple Choice	17
Grading Morale	1
Total	**75**

1. Terraforming Mars [12 points]

The year is 2200, and you are working hard to make Mars habitable for humans to live. You have limited time and resources, and each year must choose to complete one of four "terraforming tasks". Specifically, each year you can either

- increase the oxygen by 1%, or
- increase the temperature by 1° Celcius, or
- create 1 artificial lake, or
- develop 1 city.

In this problem, a *terraforming plan* consists of some number of years where we have performed exactly one terraforming task each year. Two terraforming plans are *distinct* if there is at least one year where we perform a different terraforming task in the two plans (notice this means the order in which you do the activities matters).

(a) How many distinct 25 year terraforming plans are there? [4 points] **Solution:**

4^{25} (we have 4 choices each year, that is repeated for 25 years).

(b) The council has decided that in order for Mars to be considered fully habitable, we must increase oxygen 12 times, increase the temperature 20 times, create 10 artificial lakes, and develop 10 cities.

Clearly, your project will take exactly $12 + 10 + 20 + 10 = 52$ years. You would like to develop a 52 year plan that achieves the desired conditions. How many distinct 52 year plans are there that result in the desired conditions? (**Hint:** You might think about *paths* you could take to a habitable planet) [4 points]

Solution:

This is like the paths question. An equivalent question is: how many distinct strings are there of length 53 consisting of exactly 20 T's, 12 O's, 10 L's, and 10 C's.

We can treat all the characters as distinct, which gives $52!$ such strings, but we have overcounted. We need to account for the indistringuishability among individual letters. For example, the order among the 12 years we chose to raise the oxygen does not matter (since the O's in our string are indistinguishable), so we need to divide out by $12!$.

This yields the final expression:

$$\frac{52!}{20!12!10!10!}.$$

Alternative Solution: We need to choose the 12 years we raise the oxygen out of the 52 years, so there are $\binom{52}{12}$ many ways of doing that. Next, we need to choose the 20 years we raise the temperature out of the remaining $52-12 = 40$ years, of which there are $\binom{40}{20}$ choices. Finally, we must choose the 10 years we add an artificial lake out of the remaining $52 - 12 - 20 = 20$ years, of which there are $\binom{20}{10}$ choices. After all these choices, the years we develop a city are fixed. This yields the expression

$$\binom{52}{12}\binom{40}{20}\binom{20}{10}.$$

(c) It would be ideal to have all of the conditions we wanted, but Earth is dying and 52 years is simply too far in the future. You decide that you must finish in exactly 35 years. Still, we need some amount of habitability by then, so it is required that the oxygen is raised at least 6 times, the temperature is raised at least 12 times, there are at least 4 artificial lakes, and we have developed at least 3 cities.

Notice that $6 + 12 + 4 + 3 = 25$ is less than 35, so you have some flexibility. Moreover, (for this part only) the council has imposed an ordering on the activities: you must first raise oxygen (for however many years you decide), then raise the temperature, then build lakes, then build cities. How many acceptable 35-year plans are there that meet the new minimum requirements for habitability? [4 points]

Solution:

This is stars and bars. We must have $O + T + L + C = 35$, such that $O \geq 6$ and $T \geq 12$ and $L \geq 4$ and $C \geq 3$.

We have 3 bars, and $35 - 6 - 12 - 4 - 3 = 10$ stars, so our answer is

$$\binom{3+10}{3} = \binom{3+10}{10} = \binom{13}{3}.$$

2. Passbirds [12 points]

You're creating a new social media account (so you can post pictures of your pet bird) and it's time to create a password. You're told that all passwords must be exactly 12 characters long, and made up of only lowercase letters $(a, b, \dots, z)$ and digits $(0, 1, \dots, 9)$, where repeat characters are allowed. Suppose you select your password uniformly at random from the set of all possible passwords.

Recall that the English alphabet has 26 characters, and there are 10 digits.

(a) What is the probability that your password contains at most one digit? [4 points]

Solution:

Let D_0 be the event that your password has zero digits, and let D_1 be the event that it has one digit.

$$P(\text{at most one digit}) = P(\text{zero digits}) + P(\text{one digit}) = P(D_0) + P(D_1) = \frac{|D_0| + |D_1|}{|\Omega|}$$

$|\Omega| = 36^{12}$ (select an ordered list of 12 characters, with $26 + 10 = 36$ alphanumeric options for each).

$|D_0| = 26^{12}$ (select a password only using letters) and $|D_1| = \binom{12}{1}\binom{10}{1} \cdot 26^{11} = 12 \cdot 10 \cdot 26^{11}$ (pick the spot where the 1 digit will go, then pick which digit it will be, then assign the other 11 spots a letter).

So our final answer is:

$$\frac{26^{12} + 12 \cdot 10 \cdot 26^{11}}{36^{12}}$$

(b) What is the probability that your password begins with 123 or ends in 789? [4 points]

Solution:

Let A be the event that your password begins with 123, and let B be the event that your password ends in 789. We are looking for:

$$P(\text{begin with 123 or end in 789}) = P(A \cup B) = \frac{|A \cup B|}{|\Omega|}$$

By the principle of inclusion-exclusion (PIE),

$$|A \cup B| = |A| + |B| - |A \cap B| = (26 + 10)^9 + (26 + 10)^9 - (26 + 10)^6 = 2 \cdot 36^9 - 36^6$$

$$P(A \cup B) = \boxed{\frac{2 \cdot 36^9 - 36^6}{36^{12}}}$$

(c) It's the middle of the night, and your pet bird is trying to hack into your account! He's very smart, and has figured out that your password starts with two consecutive, identical letters (e.g. $aa, bb, \ldots, zz$). He will guess uniformly at random from this new set of potential passwords. What is the probability that he guesses correctly? [4 points]

Solution:

Let Ω_1 be the new updated sample space (the set of passwords that start with two consecutive and identical letters). $|\Omega_1| = \binom{26}{1} \cdot 36^{10}$ (pick the letter to occupy the first two spots, then pick the remaining 10 spots). One of these passwords will be correct, so his success probability is just:

$$P(\text{success}) = \frac{1}{|\Omega_1|} = \frac{1}{26 \cdot 36^{10}}$$

3. Achoo! [17 points]

Achoo! You just sneezed. Was it just allergies...or something else? Let:

- S be the event that you sneezed.
- A be the event you're allergic to something in the air, which happens with probability 0.25.

(a) Assume the probability of sneezing and not being allergic to anything is 0.24. What is the probability that you sneeze, if you aren't allergic to anything? [4 points]

Solution:

By def. of conditional probability,

$$\mathbb{P}\left[S \mid A^C\right] = \frac{\mathbb{P}\left[S \cap A^C\right]}{\mathbb{P}\left[A^C\right]} = \frac{0.24}{1 - 0.25} = \frac{0.24}{0.75}$$

You begin to wonder whether you have a cold.

- Let C be the event that you have a cold, which happens with probability 0.4.

Assume that the event that you're allergic to something (or not) and the event that you have a cold (or not) are **independent** to each other.

(b) What is the probability that you have a cold, but aren't allergic to something in the air?
(If you pick out the right pieces of information, this part is very quick!) [3 points]

Solution:

Since A and C are independent, $\mathbb{P}\left[C \cap A^C\right] = \mathbb{P}\left[C\right]\mathbb{P}\left[A^C\right] = 0.4 \cdot (1 - 0.25) = 0.4 \cdot 0.75 = 0.3$

In addition, you know the following facts about yourself:

- If you're allergic to something in the air, but don't have a cold, the probability of sneezing is 0.4.
- If you have a cold, but aren't allergic to anything, the probability of sneezing is 0.8.
- If you are neither allergic nor have a cold, you will never sneeze.
- If you both are allergic, and have a cold, the probability of sneezing is 0.9

(c) What is the probability that you sneeze?
Remember you **don't** have to simplify arithmetic expressions. [5 points]

Solution:

By LTP,

$$\begin{aligned}
\mathbb{P}\left[S\right] &= \mathbb{P}\left[S, A, C\right] + \mathbb{P}\left[S, A, C^C\right] + \mathbb{P}\left[S, A^C, C\right] + \mathbb{P}\left[S, A^C, C^C\right] \\
&= \mathbb{P}\left[S \mid A, C\right]\mathbb{P}\left[A, C\right] + \mathbb{P}\left[S \mid A, C^C\right]\mathbb{P}\left[A, C^C\right] + \mathbb{P}\left[S \mid A^C, C\right]\mathbb{P}\left[A^C, C\right] + \mathbb{P}\left[S \mid A^C, C^C\right]\mathbb{P}\left[A^C, C^C\right] \\
&= 0.9(0.25)(0.4) + 0.4(0.25)(1 - 0.4) + 0.8(1 - 0.25)(0.4) + 0 \\
&= 0.9(0.25)(0.4) + 0.4(0.25)(0.6) + 0.8(0.75)(0.4)
\end{aligned}$$

where $\mathbb{P}\left[A, C\right] = \mathbb{P}\left[A\right] \cdot \mathbb{P}\left[C\right]$ by independence (and similarly for the other terms).

(d) What is the probability that you have a cold and aren't allergic to something in the air, given that you sneezed? First, fill in the blank with the appropriate notation, then compute the probability. You may use lowercase letters to refer to the answers (e.g., b to represent the answer to part (b)). [5 points]

Solution:

Using Bayes' rule,

$$\begin{aligned}\mathbb{P}\left[C \cap A^C | S\right] &= \frac{\mathbb{P}\left[S | C \cap A^C\right] \mathbb{P}\left[C \cap A^C\right]}{\mathbb{P}\left[S\right]} \\ &= \frac{0.8 \cdot b}{c}\end{aligned}$$

4. A Duck Walks up to a Lemonade Stand [16 points]

Each day, a duck walks up to a lemonade stand, and he says to the man, running the stand: "hey ... got any grapes?" Each day, the man gets annoyed that the duck is asking about grapes (when he is clearly just selling lemonade) independently with probability 0.3.

(a) Suppose this happens for n days total. How many days do we expect the man running the stand to **not** get annoyed? [3 points]

Solution:

$n(1-0.3)$.

(b) Suppose the duck gets very sad if the man got annoyed with him both that day and the day before. Over the course of n days, how many times do we expect the duck to get very sad? Explicitly define any random variables you use in this part. [5 points]

Solution:

Define X_i to be the indicator that the man got annoyed with the duck on days i and $i-1$, for $i \in \{2, \dots, n\}$. We care about X, the number of days the duck is very sad. We have that

$$X = \sum_i X_i.$$

And therefore, by LOE, $\mathbb{E}[X] = \sum_i \mathbb{E}[X_i]$.

Finally, we have $\mathbb{E}[X_i] = \text{Pr}[X_i = 1] = p^2$ So $\mathbb{E}[X] = (n-1)p^2$.

(c) Suppose now there are 2 lemonade stands, and each day the duck visits exactly one of the two stands, uniformly at random (and the duck's choice is independent each day). The man at the first stand gets annoyed with the duck's question with probability 0.3 (independently each day), and the man at the second stand gets annoyed with the duck's question with probability 0.7 (independently each day).

Let X be the number of times that someone gets annoyed with the duck over n days.
Compute the PMF p_X for X; make sure to specify what value it takes for all possible inputs. [4 points]

Solution:

X is Binomial with parameters n and $p = (p_1 + p_2)/2 = 0.5$.

(d) Suppose the man selling lemonade sells about 20 per hour, each making $\$2$. He decides to model the number of lemonades he sells per hour as a Poi(20) random variable. Supposing he works 4 hours in a day, what is the variance of his daily **profit**? Recall that if $Y \sim \text{Poi}(\lambda)$, then $\text{Var}(Y) = \lambda$. [4 points]

Solution:

Let Y_i be the number of lemonades sold in the i-th hour. Let Y be the number of lemonades sold, so $Y = Y_1 + Y_2 + Y_3 + Y_4$ By independence, $\text{Var}(Y) = \text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3) + \text{Var}(Y_4) = 4 \cdot 20$. Our profit is $2Y$, so the variance of our profit is $\text{Var}(2Y) = 2^2\text{Var}(Y) = 2^2 \cdot 4 \cdot 20$.

5. Multiple Choice [17 points]

For the questions below,

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

(a) Suppose we know A and B are mutually exclusive (i.e., disjoint) events, with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then A and B are independent. [2 points]

◯ Always
◯ Sometimes
◯ Never

Solution:

never

(b) Let $A, B \subseteq \Omega$. Then $\mathbb{P}[A \cup B] = \frac{|A|+|B|-|A \cap B|}{|\Omega|}$. [2 points]

◯ Always
◯ Sometimes
◯ Never

Solution:

Sometimes. Only if the probability space is uniform is this true.

(c) Let $A_1, \dots, A_n$ be mutually exclusive (i.e., disjoint) events.
Then $\mathbb{P}[B] = \mathbb{P}[B|A_1]\,\mathbb{P}[A_1] + \mathbb{P}[B|A_2]\,\mathbb{P}[A_2] + \cdots + \mathbb{P}[B|A_n]\,\mathbb{P}[A_n]$. [2 points]

◯ Always
◯ Sometimes
◯ Never

Solution:

Sometimes. Only if we also assume $\bigcup_i A_i = \Omega$ is this true.

(d) Let A and B be events. Suppose you know A, B are independent, $A, \overline{B}$ are independent, and $\overline{A}, B$ are independent. Are $\overline{A}$ and $\overline{B}$ independent? [2 points]

◯ Always
◯ Sometimes
◯ Never

Solution:

Always. Actually any one of the three assumptions is enough to conclude the last one.

(e) Suppose we would like to create a zoo. There are a total of n animals to choose from. We would like to pick a set of 3 animals to display prominently, and then choose any subset of the remainder to have in the back of the zoo. Then the number of possible zoos we can make is (select ALL that apply) [3 points]

- ☐ $\sum_{k=3}^{n} 2^k \cdot \binom{k}{3}$
- ☐ $\binom{n}{3}2^{n-3}$
- ☐ $n \cdot (n-1) \cdot (n-2) \cdot 2^{n-3}$
- ☐ $\binom{n}{3} \sum_{k=0}^{n-3} \binom{n}{k}$

Solution:

2, 4 (choose the 3 to display prominently, then any subset of the remaining)

(f) Let A, B, C be events. Which of the following are (always) true? Mark ALL that apply. [3 points]

- ☐ $\mathbb{P}\left[A|(B \cap C)\right] = \frac{\mathbb{P}[(B\cap C)|A]\cdot\mathbb{P}[A]}{\mathbb{P}[B\cap C]}$
- ☐ $\mathbb{P}\left[A|(B \cap C)\right] = \frac{\mathbb{P}[B|(A\cap C)]\cdot\mathbb{P}[A|C]}{\mathbb{P}[B|C]}$
- ☐ $\mathbb{P}\left[A|(B \cap C)\right] = \mathbb{P}\left[(A|B)|C\right] \cdot \mathbb{P}\left[C|(B|A)\right]$
- ☐ $\mathbb{P}\left[A|(B \cap C)\right] = \frac{\mathbb{P}[A\cap B\cap C]}{\mathbb{P}[A]}$

Solution:

1 and 2 (2 is bayes rule where everything is conditioned on C)

(g) Suppose I roll a fair 6-sided die, and then my friend rolls a fair k-sided die, where k was the outcome of my die roll. Let X be the outcome of my die roll, and Y be the outcome of my friend's die roll. Select ALL that apply. [3 points]

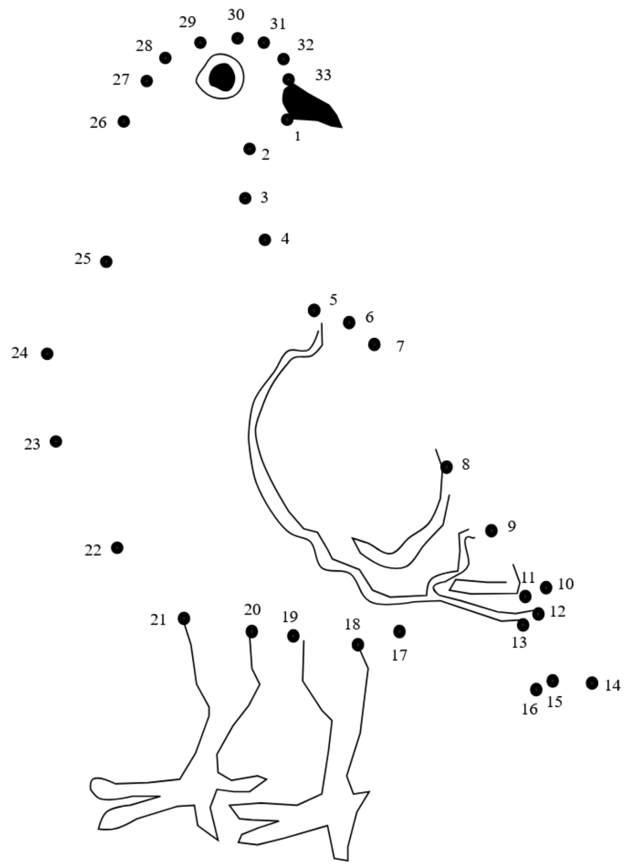
- ☐ X and Y are independent
- ☐ X and Y are mutually exclusive
- ☐ $\mathbb{E}\left[X+Y\right] = \mathbb{E}\left[X\right] + \mathbb{E}\left[Y\right]$
- ☐ $\text{Var}\left(X+Y\right) = \text{Var}\left(X\right) + \text{Var}\left(Y\right)$

Solution:

Just 3.

6. Grading Morale [1 point]

Put something on this page or finish the image below. Depending on your mood, this might be a poem, the statement of your favorite theorem, the statement of your least favorite theorem, or a piece of art. As long as you make some mark on this page, you will get the point. Looking at these helps keep the TAs happy while grading.



Additional space for any prior problem. Be sure to tell us to look here on the original problem!