# CSE 312 – Section 7

**Summer 2026**

## Review of Main Concepts

- **Multivariate: Discrete to Continuous:**

| | **Discrete** | **Continuous** |
|---|---|---|
| **Joint PMF/PDF** | $p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$ | $f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$ |
| **Joint range/support** $\Omega_{X,Y}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$ |
| **Joint CDF** | $F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$ | $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$ |
| **Normalization** | $\sum_{x,y} p_{X,Y}(x,y) = 1$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$ |
| **Marginal PMF/PDF** | $p_X(x) = \sum_y p_{X,Y}(x,y)$ | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$ |
| **Expectation** | $\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$ | $\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$ |
| **Independence** must have | $\forall x,y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ | $\forall x,y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ |

- **Law of Total Probability (RV Version):** If $X$ is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx \qquad \text{continuous } X$$

- **Conditional Expectation:** The expected value of random variable $X$ given that event $A$ has occurred, written $\mathbb{E}[X|A]$, is defined as

$$\mathbb{E}[X|A] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x | A).$$

- **Law of Total Expectation (Event Version):** Let events $A_1 \dots, A_n$ partition the sample space. Then

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i] \mathbb{P}(A_i).$$

- **Law of Total Expectation (RV Version):** Suppose $X$ and $Y$ are random variables. Then

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X | Y = y] \cdot \mathbb{P}(Y = y). \qquad \text{discrete } Y$$

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X | Y = y] f_Y(y) dy \qquad \text{continuous } Y$$

- **Expected value of $X$ conditioned on r.v. $Y$:** Suppose that $Y$ is a random variable that takes values $y_1, \dots, y_k$. Then $\mathbb{E}[X|Y]$ is the following random variable

$$\mathbb{E}[X|Y] = \begin{cases} \mathbb{E}[X|Y = y_1] & \text{with probability } \mathbb{P}(Y = y_1) \\ \mathbb{E}[X|Y = y_2] & \text{with probability } \mathbb{P}(Y = y_2) \\ \dots \\ \mathbb{E}[X|Y = y_k] & \text{with probability } \mathbb{P}(Y = y_k) \end{cases}$$

- **Law of total expectation (rewritten):** Given the above definition, we can write

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \sum_{i=1}^{k} \mathbb{E}[X|Y = y_i] \cdot \mathbb{P}(Y = y_i).$$

- **Covariance:** We may not get to this in class. To find out more, check out section 5.4 in the Tsun book. The definition: For any two random variables $X, Y$ the *covariance* defined as

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

- **Conditional distributions:**

| | **Discrete** | **Continuous** |
|---|---|---|
| **Conditional PMF/PDF** | $p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$ | $f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$ |
| **Conditional Expectation** | $\mathbb{E}[X\|Y = y] = \sum_x x p_{X\|Y}(x\|y)$ | $\mathbb{E}[X\|Y = y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y)dx$ |

- **Central Limit Theorem Problems**:

Here's a general template for solving CLT problem! Sometimes we'll be trying to solve for the probability of something (e.g., $P(X \le 10)$, and sometimes, we'll be trying to find a value of some parameter that will allow for the probability to be in a certain range (e.g., $P(X \le 10) \le 0.2$). Regardless, we still will want to apply CLT on X, and follow the same process (the only difference is that we may be solving for different things).

a) Setup the problem - write event you are interested in, in terms of sum of random variables. (what do we want to solve for/what is the probability we want to be true?)
   - Write the random variable we're interested in as a sum of i.i.d., random variables.
   - Apply CLT to $X = \sum_{i=1}^{n} X_i$ approximating it by $Y \sim N(\mu, \sigma^2)$.
   - Write the probability we're interested in.

b) *If the RVs are discrete*, apply continuity correction.

c) Normalize RV to have mean 0 and standard deviation 1: $Z = \frac{Y-\mu}{\sigma}$

d) Replace RV in probability expression with $Z \sim N(0, 1)$

e) Write in terms of $\Phi(z) = P(Z \le z)$

f) Look up in the Phi table (or do a reverse Phi table lookup if we're looking for a value of $z$ that gives us a certain probability)

## Plan for Section

- Content Review (Problem 1)
- Joint PMF's (Problem 2, 6) - do them fast.
- 3 points on a line - Problem 7
- CLT Problem 10 - review for quiz.
- Min and max of i.i.d. random variables - Problem 8 if time permits

We recommend that students look at the final problem for examples of how to set up the ranges of integration. Might be helpful on the homework.

## 1 Content Review

a) Select one: Suppose we have $n$ independent and identically distributed random variables $X_1, X_2, \ldots, X_n$, each with mean $\mu$ and variance $\sigma^2$. Let $X = \sum_{i=1}^{n} X_i$. Then as $n$ grows large, the Central Limit Theorem tells us that $X$ behaves similarly to which normal distribution?

○ $X \sim \mathcal{N}(n\mu, n\sigma^2)$
○ $X \sim \mathcal{N}(\mu, n\sigma^2)$
○ $X \sim \mathcal{N}(n\mu, \sigma^2)$
○ $X \sim \mathcal{N}(n\mu, n^2\sigma^2)$

b) **Marginal PDF.** Let $X$ and $Y$ be continuous random variables with joint PDF $f_{X,Y}(x, y)$. Which of the following correctly expresses the marginal PDF $f_X(x)$?

○ $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$
○ $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
○ $\int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t, s)\, ds\, dt$

c) **Independence and Support.** True or False: If the joint support $\Omega_{X,Y}$ of the random variables $(X, Y)$ is a circle defined by $x^2 + y^2 \leq 1$, and $\Omega_X = \Omega_Y = [0, 1]$ then $X$ and $Y$ are independent.

○ True
○ False

d) **Continuous Law of Total Probability.** Let $A$ be an event and $X$ be a continuous random variable with PDF $f_X(x)$. Which of the following is the correct expression for the Continuous Law of Total Probability?

○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x)\, dx$
○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \cap X = x) f_X(x)\, dx$
○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(X = x \mid A)\mathbb{P}(A)\, dx$

- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x) f_X(x)\, dx$

e) Select all the true statements. Suppose that $X_1, X_2, \dots, X_{30}$ are i.i.d. random variables, and let $X = \sum_{i=1}^{30} X_i$. Let $Y$ denote the normal random variable used to approximate $X$ via the CLT. Then:

- ○ If $X_i \sim \text{Ber}(p)$, then we use continuity correction to approximate

$$\mathbb{P}(12 \leq X \leq 20) \approx \mathbb{P}(11.5 \leq Y \leq 20.5)$$

- ○ If $X_i \sim \text{Exp}(\lambda)$, then we use continuity correction to approximate

$$\mathbb{P}(12 \leq X \leq 20) \approx \mathbb{P}(11.5 \leq Y \leq 20.5)$$

- ○ If $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 2) = \frac{1}{2}$, then we use continuity correction to approximate

$$\mathbb{P}(12 \leq X \leq 20) \approx \mathbb{P}(11 \leq Y \leq 21)$$

- ○ If $X_i \sim \text{Poi}(\lambda)$, then we use continuity correction to approximate

$$\mathbb{P}(12 < X < 20) \approx \mathbb{P}(12.5 \leq Y \leq 19.5)$$

**Hint:** Carefully consider which values of X satisfy the constraint on the LHS.

## 2 Joint PMF's

Suppose $X$ and $Y$ have the following joint PMF:

| X/Y | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 0 | 0.2 | 0.1 |
| 1 | 0.3 | 0 | 0.4 |

a) Identify the range of $X$ ($\Omega_X$), the range of $Y$ ($\Omega_Y$), and their joint range ($\Omega_{X,Y}$).

b) Find the marginal PMF for $X$, $p_X(x)$ for $x \in \Omega_X$.

c) Find the marginal PMF for $Y$, $p_Y(y)$ for $y \in \Omega_Y$.

d) Are $X$ and $Y$ independent? Why or why not?

e) Find $\mathbb{E}\left[X^3 Y\right]$.

## 3 Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of $n$ independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let $X_i$ be the number of times outcome $i$ occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

## 4 Do You “Urn” to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the $i$-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

a) $X_1, X_2$

b) $X_1, X_2, X_3$

## 5 Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability $p$. Let $X_1$ be the number of failures preceding the first success, and let $X_2$ be the number of failures between the first 2 successes. Find the joint pmf of $X_1$ and $X_2$. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum.

## 6 Continuous joint density

The joint density of $X$ and $Y$ is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of $W$ and $V$ is given by

$$f_{W,V}(w, v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent? Are $W$ and $V$ independent?

## 7 3 points on a line

Three values $X_1, X_2, X_3$ are selected uniformly at random, each between 0 and 1 (continuous independent uniform distributions). What is the probability that $X_2$ is greater than $X_1$ but less than $X_3$?

## 8 Min and max of i.i.d. random variables

Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \dots, X_n)$ and let $Z = \max(X_1, \dots, X_n)$. Show how to write the CDF and pdf of $Y$ and $Z$ in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

## 9 Law of Total Probability

a) Suppose we flip a coin with probability $U$ of heads, where $U$ is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ (notice this set has size $n + 1$). Let $H$ be the event that the coin comes up heads. What is $\mathbb{P}(H)$?

b) Now suppose $U \sim \mathsf{Uniform(0,1)}$ has the *continuous* uniform distribution over the interval $[0,1]$. Use the continuous law of total probability to handle this case.

c) Suppose that $X_1$ and $X_2$ are independent continuous random variables. Find an expression for $\mathbb{P}(X_1 < 2X_2)$ using the law of total probability, in terms of $F_{X_1}, F_{X_2}, f_{X_1}, f_{X_2}$. (Your answer will be in the form of a single integral, and requires no calculations – do not evaluate it).

d) Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Find $s$, where $\Phi(s) = \mathbb{P}(X_1 < 2X_2)$ using the fact that linear combinations of independent normal random variables are still normal.

e) Suppose $Z = X + Y$, where $X$ and $Y$ are independent. $Z$ is called the *convolution* of the two random variables. If $X, Y, Z$ are discrete, using the law of total probability, we can write

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_x \mathbb{P}(X = x \cap Y = z - x) = \sum_x p_X(x)\, p_Y(z - x)$$

Write an analogous expression for $F_Z(z)$ in the case that $X, Y, Z$ are continuous where, again, $X$ and $Y$ are independent.

## 10 Bad Computer

Each day, the probability your computer crashes is 10%, independent of every other day. Suppose we want to evaluate the computer's performance over the next 100 days.

a) Let $X$ be the number of crash-free days in the next 100 days. What distribution does $X$ have? Identify $\mathbb{E}[X]$ and $Var(X)$ as well. Write an exact expression for $\mathbb{P}(X \geq 87)$.

b) Approximate the probability of at least 87 crash-free days out of the next 100 days using the Central Limit Theorem.

## 11 Jointly distributed random variables involving 3 variables

a) **Validating a Joint Density.** Let $X, Y$, and $Z$ be continuous random variables. To verify that $f_{X,Y,Z}(x, y, z) = 6$ for the region $0 \leq x \leq y \leq z \leq 1$ (and 0 otherwise) is a valid joint probability density function, which of the following equations must hold true?

- ○ $\int_0^1 \int_0^1 \int_0^1 6\, dx\, dy\, dz = 1$
- ○ $\int_0^1 \int_0^z \int_0^y 6\, dx\, dy\, dz = 1$
- ○ $\int_0^1 \int_0^x \int_0^y 6\, dz\, dy\, dx = 1$
- ○ $\int_0^1 \int_x^1 \int_y^1 6\, dx\, dy\, dz = 1$

b) **Integrating out a Variable.** Let $X, Y$, and $Z$ be continuous random variables with joint PDF $f_{X,Y,Z}(x, y, z) = 6$ for the region $0 \leq x \leq y \leq z \leq 1$, and 0 otherwise. Which of the following correctly expresses the joint marginal PDF $f_{X,Y}(x, y)$ for the valid region?

- ○ $\int_0^1 6\, dz$
- ○ $\int_x^y 6\, dz$

- ○ $\int_0^y 6\,dz$
- ○ $\int_y^1 6\,dz$

c) **The 3D Simplex.** Let $X, Y$, and $Z$ be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes the probability that $X+Y+Z \le 1$?

- ○ $\int_0^1 \int_0^1 \int_0^1 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^x \int_0^y 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^{1-x} \int_0^1 1\,dz\,dy\,dx$

d) **Bounding with Max.** Let $X, Y$, and $Z$ be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes $\mathbb{P}(X \ge \max(Y, Z))$?

- ○ $\int_0^1 \int_x^1 \int_x^1 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^1 \int_0^{\max(y,z)} 1\,dx\,dy\,dz$
- ○ $\int_0^1 \int_0^x \int_0^x 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^1 \int_{\min(y,z)}^1 1\,dx\,dy\,dz$

e) **Conditional PDF.** (Not covered in class.) For two continuous random variables $X$ and $Y$, which of the following defines the conditional PDF $f_{X|Y}(x|y)$?

- ○ $\frac{f_{X,Y}(x,y)}{f_X(x)}$
- ○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
- ○ $f_X(x) f_Y(y)$
- ○ $\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$