# CSE 312 – Section 7 Solutions

## Summer 2026

## Review of Main Concepts

- **Multivariate: Discrete to Continuous:**

| | **Discrete** | **Continuous** |
|---|---|---|
| **Joint PMF/PDF** | $p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$ | $f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$ |
| **Joint range/support** $\Omega_{X,Y}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$ |
| **Joint CDF** | $F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$ | $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$ |
| **Normalization** | $\sum_{x,y} p_{X,Y}(x,y) = 1$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$ |
| **Marginal PMF/PDF** | $p_X(x) = \sum_y p_{X,Y}(x,y)$ | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$ |
| **Expectation** | $\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$ | $\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$ |
| **Independence** must have | $\forall x,y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ | $\forall x,y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ |

- **Law of Total Probability (RV Version):** If $X$ is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A|X=x) p_X(x) \qquad \text{discrete } X$$

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A|X=x) f_X(x) dx \qquad \text{continuous } X$$

- **Conditional Expectation:** The expected value of random variable $X$ given that event $A$ has occurred, written $\mathbb{E}[X|A]$, is defined as

$$\mathbb{E}[X|A] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X=x|A).$$

- **Law of Total Expectation (Event Version):** Let events $A_1 \dots, A_n$ partition the sample space. Then

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i] \mathbb{P}(A_i).$$

- **Law of Total Expectation (RV Version):** Suppose $X$ and $Y$ are random variables. Then

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X|Y=y] \cdot \mathbb{P}(Y=y). \qquad \text{discrete } Y$$

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X|Y=y] f_Y(y) dy \qquad \text{continuous } Y$$

- **Expected value of $X$ conditioned on r.v. $Y$:** Suppose that $Y$ is a random variable that takes values $y_1, \dots, y_k$. Then $\mathbb{E}[X|Y]$ is the following random variable

$$\mathbb{E}[X|Y] = \begin{cases} \mathbb{E}[X|Y = y_1] & \text{with probability } \mathbb{P}(Y = y_1) \\ \mathbb{E}[X|Y = y_2] & \text{with probability } \mathbb{P}(Y = y_2) \\ \dots & \\ \mathbb{E}[X|Y = y_k] & \text{with probability } \mathbb{P}(Y = y_k) \end{cases}$$

- **Law of total expectation (rewritten):** Given the above definition, we can write

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \sum_{i=1}^{k} \mathbb{E}[X|Y = y_i] \cdot \mathbb{P}(Y = y_i).$$

- **Covariance:** We may not get to this in class. To find out more, check out section 5.4 in the Tsun book. The definition: For any two random variables $X, Y$ the *covariance* defined as

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

- **Conditional distributions:**

| | **Discrete** | **Continuous** |
|---|---|---|
| **Conditional PMF/PDF** | $p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$ | $f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$ |
| **Conditional Expectation** | $\mathbb{E}[X\|Y = y] = \sum_x x p_{X\|Y}(x\|y)$ | $\mathbb{E}[X\|Y = y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$ |

- **Central Limit Theorem Problems**:

Here's a general template for solving CLT problem! Sometimes we'll be trying to solve for the probability of something (e.g., $P(X \leq 10)$, and sometimes, we'll be trying to find a value of some parameter that will allow for the probability to be in a certain range (e.g., $P(X \leq 10) \leq 0.2$). Regardless, we still will want to apply CLT on X, and follow the same process (the only difference is that we may be solving for different things).

a) Setup the problem - write event you are interested in, in terms of sum of random variables. (what do we want to solve for/what is the probability we want to be true?)
   - Write the random variable we're interested in as a sum of i.i.d., random variables.
   - Apply CLT to $X = \sum_{i=1}^{n} X_i$ approximating it by $Y \sim N(\mu, \sigma^2)$.
   - Write the probability we're interested in.

b) *If the RVs are discrete*, apply continuity correction.

c) Normalize RV to have mean 0 and standard deviation 1: $Z = \frac{Y-\mu}{\sigma}$

d) Replace RV in probability expression with $Z \sim N(0, 1)$

e) Write in terms of $\Phi(z) = P(Z \leq z)$

f) Look up in the Phi table (or do a reverse Phi table lookup if we're looking for a value of $z$ that gives us a certain probability)

## Plan for Section

- Content Review (Problem 1)
- Joint PMF's (Problem 2, 6) - do them fast.
- 3 points on a line - Problem 7
- CLT Problem 10 - review for quiz.
- Min and max of i.i.d. random variables - Problem 8 if time permits

We recommend that students look at the final problem for examples of how to set up the ranges of integration. Might be helpful on the homework.

## 1 Content Review

a) Select one: Suppose we have $n$ independent and identically distributed random variables $X_1, X_2, \ldots, X_n$, each with mean $\mu$ and variance $\sigma^2$. Let $X = \sum_{i=1}^{n} X_i$. Then as $n$ grows large, the Central Limit Theorem tells us that $X$ behaves similarly to which normal distribution?

○ $X \sim \mathcal{N}(n\mu, n\sigma^2)$
○ $X \sim \mathcal{N}(\mu, n\sigma^2)$
○ $X \sim \mathcal{N}(n\mu, \sigma^2)$
○ $X \sim \mathcal{N}(n\mu, n^2\sigma^2)$

The first one. By linearity of expectation, $X = n\mu$. Now since each of the rvs are independent, we may say that $\text{Var}(X) = n\sigma^2$. Then as $n$ grows large, $X$ behaves similarly to a normal random variable with the same expectation and variance as itself.

b) **Marginal PDF.** Let $X$ and $Y$ be continuous random variables with joint PDF $f_{X,Y}(x, y)$. Which of the following correctly expresses the marginal PDF $f_X(x)$?

○ $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$
○ $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
○ $\int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t, s)\, ds\, dt$

**Answer:** $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
To find the marginal distribution of $X$, you must "integrate out" the other variable, $Y$, over its entire range. *(Note: The first option incorrectly integrates out $x$, which would leave a function of $y$. The third option is the conditional PDF $f_{X|Y}(x|y)$, and the fourth option is the joint CDF $F_{X,Y}(x, y)$.)*

c) **Independence and Support.** True or False: If the joint support $\Omega_{X,Y}$ of the random variables $(X, Y)$ is a circle defined by $x^2 + y^2 \leq 1$, and $\Omega_X = \Omega_Y = [0, 1]$ then $X$ and $Y$ are independent.

- ○ True
- ○ False

**Answer: False**
For $X$ and $Y$ to be independent, their joint support must be the Cartesian product of their marginal supports ($\Omega_{X,Y} = \Omega_X \times \Omega_Y$). A circle is not the Cartesian product of $[0, 1]$.

d) **Continuous Law of Total Probability.** Let $A$ be an event and $X$ be a continuous random variable with PDF $f_X(x)$. Which of the following is the correct expression for the Continuous Law of Total Probability?

- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x)\, dx$
- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \cap X = x) f_X(x)\, dx$
- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(X = x \mid A)\mathbb{P}(A)\, dx$
- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x) f_X(x)\, dx$

**Answer:** $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x) f_X(x)\, dx$
The Continuous Law of Total Probability scales the conditional probability of $A$ given $X = x$ by the density of $X$ at $x$, integrated over all possible values of $X$. *(Note: Option 1 is missing the density weighting. Option 2 incorrectly combines an intersection with the density function. Option 3 is a jumbled application of Bayes' rule elements.)*

e) Select all the true statements. Suppose that $X_1, X_2, \ldots, X_{30}$ are i.i.d. random variables, and let $X = \sum_{i=1}^{30} X_i$. Let $Y$ denote the normal random variable used to approximate $X$ via the CLT. Then:

- ○ If $X_i \sim \text{Ber}(p)$, then we use continuity correction to approximate
$$\mathbb{P}(12 \leq X \leq 20) \approx \mathbb{P}(11.5 \leq Y \leq 20.5)$$
- ○ If $X_i \sim \text{Exp}(\lambda)$, then we use continuity correction to approximate
$$\mathbb{P}(12 \leq X \leq 20) \approx \mathbb{P}(11.5 \leq Y \leq 20.5)$$
- ○ If $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 2) = \frac{1}{2}$, then we use continuity correction to approximate
$$\mathbb{P}(12 \leq X \leq 20) \approx \mathbb{P}(11 \leq Y \leq 21)$$
- ○ If $X_i \sim \text{Poi}(\lambda)$, then we use continuity correction to approximate
$$\mathbb{P}(12 < X < 20) \approx \mathbb{P}(12.5 \leq Y \leq 19.5)$$

**Hint:** Carefully consider which values of X satisfy the constraint on the LHS.

- First is **true**: $X$ takes integer values with spacing 1, so the endpoints are expanded by (0.5).
- Second is **false**: $X$ is continuous, so no continuity correction is needed.
- Third is **true**: $X$ takes even integer values with spacing 2, so the endpoints are expanded by 1 instead of 0.5.
- Fourth is **true**: since $X$ is integer-valued, $\{12 < X < 20\} = \{13 \leq X \leq 19\}$, giving the corrected interval $12.5 \leq Y \leq 19.5$.

## 2 Joint PMF's

Suppose $X$ and $Y$ have the following joint PMF:

| X/Y | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 0 | 0.2 | 0.1 |
| 1 | 0.3 | 0 | 0.4 |

a) Identify the range of $X$ ($\Omega_X$), the range of $Y$ ($\Omega_Y$), and their joint range ($\Omega_{X,Y}$).

$\Omega_X = \{0, 1\}$, $\Omega_Y = \{1, 2, 3\}$, and $\Omega_{X,Y} = \{(0, 2), (0, 3), (1, 1), (1, 3)\}$

b) Find the marginal PMF for $X$, $p_X(x)$ for $x \in \Omega_X$.

$$p_X(0) = \sum_y p_{X,Y}(0, y) = 0 + 0.2 + 0.1 = 0.3$$
$$p_X(1) = 1 - p_X(0) = 0.7$$

c) Find the marginal PMF for $Y$, $p_Y(y)$ for $y \in \Omega_Y$.

$$p_Y(1) = \sum_x p_{X,Y}(x, 1) = 0 + 0.3 = 0.3$$
$$p_Y(2) = \sum_x p_{X,Y}(x, 2) = 0.2 + 0 = 0.2$$
$$p_Y(3) = \sum_x p_{X,Y}(x, 3) = 0.1 + 0.4 = 0.5$$

d) Are $X$ and $Y$ independent? Why or why not?

No, since a necessary condition is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

e) Find $\mathbb{E}\left[X^3 Y\right]$.

Note that $X^3 = X$ since $X$ takes values in $\{0, 1\}$.

$$\mathbb{E}\left[X^3 Y\right] = \mathbb{E}\left[XY\right] = \sum_{(x,y)\in\Omega_{X,Y}} xy p_{X,Y}(x, y) = 1 \cdot 1 \cdot 0.3 + 1 \cdot 3 \cdot 0.4 = 1.5$$

## 3 Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of $n$ independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let $X_i$ be the number of times outcome $i$ occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

We use a similar argument as for the binomial PMF. $\binom{n}{x_1,x_2,x_3}$ is the number of ways to select which of the $n$ outcomes result in each of the 3 outcomes. Then, we multiply the probabilities of each trial being the corresponding outcome (e.g., $p_1^{x_1}$ is the probability that all $x_1$ trials end up being outcome 1). This gives use the following PMF:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

## 4 Do You "Urn" to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the $i$-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

a) $X_1, X_2$

Here is one way of defining the joint pmf of $X_1, X_2$

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156}$$

$$\mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{8}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 1) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 1 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{5}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156}$$

b) $X_1, X_2, X_3$

Instead of listing out all the individual probabilities, we could write a more compact formula for the pmf. In this problem, the denominator is always $P(13, k)$, where $k$ is the number of random variables in the joint pmf. And the numerator is $P(5, i)$ times $P(8, j)$ where $i$ and $j$ are the number of 1s and 0s, respectively. If we wish to compute $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$, then the number of 1s (i.e., white balls) is $x_1 + x_2 + x_3$, and the number of 0s (i.e., red balls) is $(1 - x_1) + (1 - x_2) + (1 - x_3)$. Then, we can write the pmf as follows:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{10!}{13!} \cdot \frac{5!}{(5 - x_1 - x_2 - x_3)!} \cdot \frac{8!}{(5 + x_1 + x_2 + x_3)!}$$

## 5 Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability $p$. Let $X_1$ be the number of failures preceding the first success, and let $X_2$ be the number of failures between the first 2 successes. Find the joint pmf of $X_1$ and $X_2$. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum.

$X_1$ and $X_2$ take on two particular values $x_1$ and $x_2$, when there are $x_1$ failures followed by one success, and then $x_2$ failures followed by one success. Since the Bernoulli trials are independent the joint pmf is

$$p_{X_1,X_2}(x_1, x_2) = (1 - p)^{x_1} p \cdot (1 - p)^{x_2} p = (1 - p)^{x_1+x_2} p^2$$

for $(x_1, x_2) \in \Omega_{X_1,X_2} = \{0, 1, 2, ...\} \times \{0, 1, 2, ...\}$ By the definition of expectation

$$E[\sqrt{X_1 X_2}] = \sum_{(x_1,x_2)\in\Omega_{X_1,X_2}} \sqrt{x_1 x_2} \cdot (1 - p)^{x_1+x_2} p^2.$$

## 6 Continuous joint density

The joint density of $X$ and $Y$ is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of $W$ and $V$ is given by

$$f_{W,V}(w, v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent? Are $W$ and $V$ independent?

For two random variables $X, Y$ to be independent, we must have $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X,\ y \in \Omega_Y$. Let's start with $X$ and $Y$ by finding their marginal PDFs. By definition, and using the fact that the joint PDF is 0 outside of $y > 0$, we get:

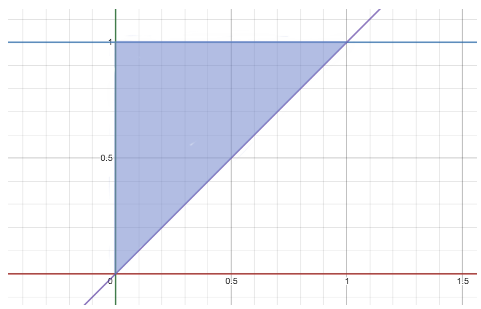
$$f_X(x) = \int_0^\infty xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of $Y$, again over the range $x > 0$:

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x, y > 0$, $X$ and $Y$ are independent.

We can see that $W$ and $V$ are not independent simply by observing that $\Omega_W = (0, 1)$ and $\Omega_V = (0, 1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product. Specifically, looking at their range of $f_{W,V}(w, v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we see that :




The shaded area is where the joint pdf is strictly positive. Looking at it, we can see that it is not rectangular, and therefore it is not the case that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the Cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.

## 7 3 points on a line

Three values $X_1, X_2, X_3$ are selected uniformly at random, each between 0 and 1 (continuous independent uniform distributions). What is the probability that $X_2$ is greater than $X_1$ but less than $X_3$?

There are mutliple ways to approach this problem! Let $X_1, X_2, X_3 \sim Unif(0, 1)$. We want to find $\mathbb{P}(X_1 < X_2 < X_3)$.

**Method 1: Condition on $X_2$.**

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < X_2 < X_3 \mid X_2 = x)\ f_{X_2}(x)\ dx && \text{Continuous LoTP} \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x | X_2 = x)\ f_{X_2}(x)\ dx \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x)\ f_{X_2}(x)\ dx && \text{Indep of } X_1, X_3 \text{ from } X_2 \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x)\ \mathbb{P}(x < X_3)\ f_{X_2}(x)\ dx && \text{Indep of } X_1, X_3 \\
&= \int_{-\infty}^{\infty} F_{X_1}(x)\ (1 - F_{X_3}(x))\ f_{X_2}(x)\ dx \\
&= \int_0^1 x\ (1-x)\ 1\ dx \\
&= \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 = \frac{1}{6}
\end{aligned}$$

**Method 2: Integrate over each variable.** First, find the joint PDF of the three variables using independence

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) = \begin{cases} 1 & \text{if} \quad 0 \le x_1, x_2, x_3 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

So, the probability of the event $X_1 < X_2 < X_3$ is exactly the volume of the region where $x_1 < x_2 < x_2$. In order to integrate over these region, start with $x_2$ (this has a range of 0 to 1), then $x_1$ (since $x_1 < x_2$, its range is 0 to $x_2$), and finally $x_3$ (since $x_2 < x_3 < 1$, its range is $x_2$ to 1).

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_0^1 \int_0^{x_2} \int_{x_2}^1 f_{X_1,X_2,X_3}(X_1, X_2, X_3)\ dx_3 dx_1 dx_2 \\
&= \int_0^1 \int_0^{x_2} \int_{x_2}^1 1\ dx_3 dx_1 dx_2 \\
&= \int_0^1 \int_0^{x_2} x_3 \bigg|_{x_2}^1 dx_1 dx_2 = \int_0^1 \int_0^{x_2} 1 - x_2\ dx_1 dx_2 \\
&= \int_0^1 (1 - x_2) x_1 \bigg|_0^{x_2} dx_2 = \int_0^1 (1 - x_2) x_2\ dx_2 \\
&= \frac{x^2}{2} - \frac{x^3}{2}\bigg|_0^1 = \frac{1}{6}
\end{aligned}$$

**Method 3: Identify symmetry.** The key idea is to note that since $X_1, X_2, X_3$ are i.i.d, every ordered sequence is equally likely, i.e $P(X_1 < X_2 < X_3) = P(X_1 < X_3 < X_2) = \cdots =$

$P(X_3 < X_2 < X_1)$.

Since there are 3 variables, there are $3! = 6$ equally likely sequences. Also note that the 6 events are disjoint (no set of them can be occur simultaneously) and it is impossible for none of them to occur (ties like $X_1 = X_2$ have probability 0). Therefore, they partition the sample space, so their probabilities should add up to 1.

$$\begin{aligned}
1 &= P(X_1 < X_2 < X_3) + \cdots + P(X_3 < X_2 < X_1) && \text{Events partition the sample space} \\
&= 6 \cdot P(X_1 < X_2 < X_3) && \text{Equally likely sequences} \\
&\implies P(X_1 < X_2 < X_3) = \frac{1}{6}
\end{aligned}$$

## 8 Min and max of i.i.d. random variables

Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \dots, X_n)$ and let $Z = \max(X_1, \dots, X_n)$. Show how to write the CDF and pdf of $Y$ and $Z$ in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

First we compute the CDFs of $Z$ and $Y$ as follows:

$$\begin{aligned}
F_Z(z) &= P(Z < z) \\
&= P(X_1 < z, ..., X_n < z) && \text{[Definition of max]} \\
&= P(X_1 < z) \cdot ... \cdot P(X_n < z) && \text{[Independence]} \\
&= (F_X(z))^n
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= P(Y < y) \\
&= 1 - P(Y > y) \\
&= 1 - P(X_1 > y, ..., X_n > y) && \text{[Definition of min]} \\
&= 1 - P(X_1 > y) \cdot ... \cdot P(X_n > y) && \text{[Independence]} \\
&= 1 - (1 - F_X(y))^n
\end{aligned}$$

Using the fact that $f_X(x) = \frac{d}{dx} F_X(x)$ and the CDFs that we found we can compute the pdfs of $Z$ and $Y$ as follows:

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \frac{d}{dz} (F_X(z))^n \\
&= n \cdot F_X(z)^{n-1} \cdot \left( \frac{d}{dz} F_X(z) \right) \\
&= n \cdot F_X(z)^{n-1} \cdot f_X(z)
\end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy}\left(1-(1-F_X(y))^n\right) \\
&= -n \cdot (1-F_X(y))^{n-1} \cdot \frac{d}{dy}(1-F_X(y)) \\
&= n \cdot (1-F_X(y))^{n-1} \cdot f_X(y)
\end{aligned}
$$

## 9 Law of Total Probability

a) Suppose we flip a coin with probability $U$ of heads, where $U$ is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ (notice this set has size $n+1$). Let $H$ be the event that the coin comes up heads. What is $\mathbb{P}(H)$?

We can use the law of total probability, conditioning on $U = \frac{k}{n}$ for $k = 0, ..., n$.

$$
\begin{aligned}
\mathbb{P}(H) &= \sum_{k=0}^{n} \mathbb{P}\left(H \mid U = \frac{k}{n}\right) \mathbb{P}\left(U = \frac{k}{n}\right) = \sum_{k=0}^{n} \frac{k}{n} \cdot \frac{1}{n+1} \\
&= \frac{1}{n(n+1)} \sum_{k=0}^{n} k = \frac{1}{n(n+1)} \frac{n(n+1)}{2} = \frac{1}{2}
\end{aligned}
$$

b) Now suppose $U \sim \mathsf{Uniform(0,1)}$ has the *continuous* uniform distribution over the interval $[0,1]$. Use the continuous law of total probability to handle this case.

We can perform basically the same process as above, just using an integral instead of a sum. The values that $U$ can take on are anywhere in the continuous interval $[0,1]$, so we integrate over that with respect to $u$ and use the PDF of $U$. Plugging that in we can get the same answer of $\frac{1}{2}$ as before.

$$
\mathbb{P}(H) = \int_0^1 \mathbb{P}(H \mid U = u) f_U(u) du = \int_0^1 u \cdot 1 du = \frac{1}{2}[u^2]_0^1 = \frac{1}{2}
$$

c) Suppose that $X_1$ and $X_2$ are independent continuous random variables. Find an expression for $\mathbb{P}(X_1 < 2X_2)$ using the law of total probability, in terms of $F_{X_1}, F_{X_2}, f_{X_1}, f_{X_2}$. (Your answer will be in the form of a single integral, and requires no calculations – do not evaluate it).

We use the continuous version of the “Law of Total Probability” to integrate over all possible values of $X_2$:

$$
\mathbb{P}(X_1 < 2X_2) = \int_{-\infty}^{\infty} \mathbb{P}(X_1 < 2X_2 \mid X_2 = x_2) f_{X_2}(x_2) dx_2 = \int_{-\infty}^{\infty} F_{X_1}(2x_2) f_{X_2}(x_2)\, dx_2
$$

d) Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Find $s$, where $\Phi(s) = \mathbb{P}(X_1 < 2X_2)$ using the fact that linear combinations of independent normal random variables are still normal.

Let $X_3 = X_1 - 2X_2$, so that $X_3 \sim \mathcal{N}(\mu_1 - 2\mu_2, \sigma_1^2 + 4\sigma_2^2)$ (by the reproductive property of normal distributions)

$$\mathbb{P}(X_1 < 2X_2) = \mathbb{P}(X_1 - 2X_2 < 0) = \mathbb{P}(X_3 < 0) = \mathbb{P}\left(\frac{X_3 - (\mu_1 - 2\mu_2)}{\sqrt{\sigma_1^2 + 4\sigma_2^2}} < \frac{0 - (\mu_1 - 2\mu_2)}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}\right)$$

$$= \mathbb{P}\left(Z < \frac{2\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}\right) = \Phi\left(\frac{2\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}\right) \implies s = \frac{2\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}$$

e) Suppose $Z = X + Y$, where $X$ and $Y$ are independent. $Z$ is called the *convolution* of the two random variables. If $X, Y, Z$ are discrete, using the law of total probability, we can write

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_x \mathbb{P}(X = x \cap Y = z - x) = \sum_x p_X(x)\, p_Y(z - x)$$

Write an analogous expression for $F_Z(z)$ in the case that $X, Y, Z$ are continuous where, again, $X$ and $Y$ are independent.

$$F_Z(z) = \mathbb{P}(X+Y \le z) = \int_{-\infty}^{\infty} \mathbb{P}(Y \le z - X \mid X = x) f_X(x)dx = \int_{-\infty}^{\infty} F_Y(z - x) f_X(x)dx$$

## 10 Bad Computer

Each day, the probability your computer crashes is 10%, independent of every other day. Suppose we want to evaluate the computer's performance over the next 100 days.

a) Let $X$ be the number of crash-free days in the next 100 days. What distribution does $X$ have? Identify $\mathbb{E}[X]$ and $Var(X)$ as well. Write an exact expression for $\mathbb{P}(X \ge 87)$.

Since $X$ counts the number of crash-free days (successes) in 100 days (trials), where each trial is a success with probability 0.9, we can see that $X$ is binomial with $n = 100$ and $p = 0.9$, or $X \sim \mathsf{Binomial}(100, 0.9)$. Hence, $\mathbb{E}[X] = np = 90$ and $Var(X) = np(1 - p) = 9$. Finally,

$$\mathbb{P}(X \ge 87) = \sum_{k=87}^{100} \binom{100}{k} (0.9)^k (1 - 0.9)^{100-k}$$

b) Approximate the probability of at least 87 crash-free days out of the next 100 days using the Central Limit Theorem.

From the previous part, we know that $X = 90$ and $\text{Var}(X) = 9$.

$$\begin{aligned}\mathbb{P}(X \geq 87) &= \mathbb{P}(86.5 < X < 100.5) = \mathbb{P}(\frac{86.5-90}{3} < \frac{X-90}{3} < \frac{100.5-90}{3}) \\ &\approx \mathbb{P}(-1.17 < \frac{X-90}{3} < 3.5) \approx \Phi(3.5) + \Phi(1.17) - 1 \approx 0.8788\end{aligned}$$

Notice that, if you had used $86.5 < X$ in place of $86.5 < X < 100.5$, your answer would have been nearly the same, because $\Phi(3.5)$ is so close to 1.

## 11 Jointly distributed random variables involving 3 variables

a) **Validating a Joint Density.** Let $X, Y$, and $Z$ be continuous random variables. To verify that $f_{X,Y,Z}(x, y, z) = 6$ for the region $0 \leq x \leq y \leq z \leq 1$ (and 0 otherwise) is a valid joint probability density function, which of the following equations must hold true?

- ○ $\int_0^1 \int_0^1 \int_0^1 6\,dx\,dy\,dz = 1$
- ○ $\int_0^1 \int_0^z \int_0^y 6\,dx\,dy\,dz = 1$
- ○ $\int_0^1 \int_0^x \int_0^y 6\,dz\,dy\,dx = 1$
- ○ $\int_0^1 \int_x^1 \int_y^1 6\,dx\,dy\,dz = 1$

**Answer:** $\int_0^1 \int_0^z \int_0^y 6\,dx\,dy\,dz = 1$
To be a valid joint PDF, the integral of the density over the entire valid region must equal 1. The support is given by the chain of inequalities $0 \leq x \leq y \leq z \leq 1$. Setting up the bounds from the outside in using the order $dx\,dy\,dz$: the outermost variable $z$ ranges from absolute minimum to maximum (0 to 1). For a fixed $z$, the variable $y$ is bounded between 0 and $z$. Finally, for fixed $y$ and $z$, the innermost variable $x$ is bounded between 0 and $y$.

$$\int_0^1 \int_0^z \int_0^y 6\,dx\,dy\,dz = 1$$

b) **Integrating out a Variable.** Let $X, Y$, and $Z$ be continuous random variables with joint PDF $f_{X,Y,Z}(x, y, z) = 6$ for the region $0 \leq x \leq y \leq z \leq 1$, and 0 otherwise. Which of the following correctly expresses the joint marginal PDF $f_{X,Y}(x, y)$ for the valid region?

- ○ $\int_0^1 6\,dz$
- ○ $\int_x^y 6\,dz$
- ○ $\int_0^y 6\,dz$
- ○ $\int_y^1 6\,dz$

**Answer:** $\int_y^1 6\, dz$
To find the joint marginal PDF $f_{X,Y}(x,y)$, we must "integrate out" the variable $Z$ over its valid range. Given the support bounds $0 \le x \le y \le z \le 1$, for any fixed values of $x$ and $y$, the variable $z$ ranges from a minimum of $y$ to a maximum of 1.

$$f_{X,Y}(x,y) = \int_y^1 f_{X,Y,Z}(x,y,z)\, dz = \int_y^1 6\, dz$$

*(Note: The bounds are the key trap here. Because the joint PDF is only non-zero when $z \ge y$, the lower limit of integration must be $y$, not 0 or $x$.)*

c) **The 3D Simplex.** Let $X, Y$, and $Z$ be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes the probability that $X + Y + Z \le 1$?

- ○ $\int_0^1 \int_0^1 \int_0^1 1\, dz\, dy\, dx$
- ○ $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\, dz\, dy\, dx$
- ○ $\int_0^1 \int_0^x \int_0^y 1\, dz\, dy\, dx$
- ○ $\int_0^1 \int_0^{1-x} \int_0^1 1\, dz\, dy\, dx$

**Answer:** $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\, dz\, dy\, dx$
First, the joint density of $X, Y, Z$ is $f_{X,Y,Z}(x,y,z) = 1$ due to independence and standard uniform distributions. We need to integrate this density over the region where $X+Y+Z \le 1$. If we fix the outer bounds for $X = x$ from 0 to 1, the remaining "budget" for $Y$ is $1-x$, so $Y$ integrates from 0 to $1-x$. Finally, given $X = x$ and $Y = y$, the variable $Z$ is bounded above by $1 - x - y$.

$$\mathbb{P}(X + Y + Z \le 1) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\, dz\, dy\, dx$$

d) **Bounding with Max.** Let $X, Y$, and $Z$ be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes $\mathbb{P}(X \ge \max(Y,Z))$?

- ○ $\int_0^1 \int_x^1 \int_x^1 1\, dz\, dy\, dx$
- ○ $\int_0^1 \int_0^1 \int_0^{\max(y,z)} 1\, dx\, dy\, dz$
- ○ $\int_0^1 \int_0^x \int_0^x 1\, dz\, dy\, dx$
- ○ $\int_0^1 \int_0^1 \int_{\min(y,z)}^1 1\, dx\, dy\, dz$

**Answer:** $\int_0^1 \int_0^x \int_0^x 1\, dz\, dy\, dx$
The joint density is $f_{X,Y,Z}(x,y,z) = 1$. The condition $X \ge \max(Y,Z)$ is equivalent to stating that $X \ge Y$ **and** $X \ge Z$. It is easiest to set $x$ as the outermost integral bounding

from 0 to 1. Then, for any fixed value of $X = x$, both $Y$ and $Z$ are independently bounded to be less than or equal to $x$. Thus, $y$ ranges from 0 to $x$, and $z$ ranges from 0 to $x$.

$$\mathbb{P}(X \geq \max(Y, Z)) = \int_0^1 \int_0^x \int_0^x 1 \, dz \, dy \, dx$$

e) **Conditional PDF.** (Not covered in class.) For two continuous random variables $X$ and $Y$, which of the following defines the conditional PDF $f_{X|Y}(x|y)$?

- ○ $\frac{f_{X,Y}(x,y)}{f_X(x)}$
- ○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
- ○ $f_X(x)f_Y(y)$
- ○ $\int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy$

**Answer:** $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
The conditional PDF of $X$ given $Y = y$ is the joint PDF divided by the marginal PDF of $Y$ evaluated at $y$. *(Note: Option 1 is $f_{Y|X}(y|x)$, option 3 is the joint PDF only if X and Y are independent, and option 4 is the marginal PDF of X.)*