CSE 312 Section 3

Conditional Probability

Administrivia

Announcements & Reminders

- PSet 1 grades released on gradescope
  - check your submission to read comments
  - Regrade requests open ~24 hours after grades are released and close after a week
- PSet 2 was due yesterday 7/8 @ 11:59 PM
  - Late deadline Friday @ 11:59 PM (max of 2 late days per problem set)
- PSet 3 is now out on the course website
  - Due Wednesday 7/15 @ 11:59 PM
  - hw3 has a coding part!
  - you will be working partly in Ed, watch the video posted on Ed Lessons and read instructions on Pset!

Naive Bayes

A preview to homework this week ….

- *Make sure that you: watch the video linked in the assignment and read section 9.3 in the book **BEFORE** doing your homework*
- Start Early!
- Ask Questions on Ed!

Review

Any lingering questions from this last week?

Each week in section, we'll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we'll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Conditional Probability

Conditional Probability

$$P(A|B)$$

probability of the event A occurring
given that
the event B occurs

"what is the probability that event A happens given that event B happened?"

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

–BAYES RULE–

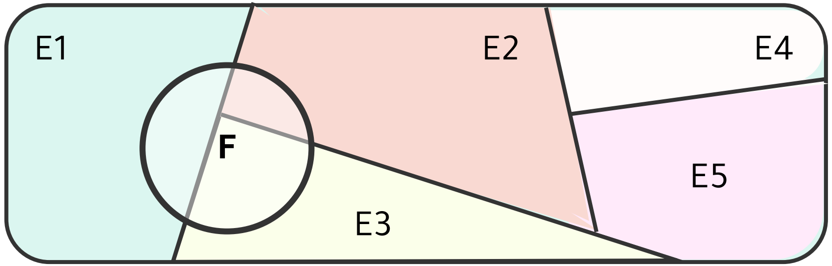
TIPS:
- start by writing all the probabilities you know
- write down what you want to find

Conditional Probability

We can use conditional probability to help calculate more complex probabilities!

Conditional Probability - LTP

we can *partition a sample space* into discrete events



$\Omega = E1 \cup E2 \cup E3 \cup ... \cup E5$

divided the set of all possible outcomes
into "disjoint" event sets

The probability of any other event F that is inside of this sample space Ω is

$$P(F) = P(F \cap E1) + P(F \cap E2) \dots + P(F \cap E5)$$

by definition of cond. probability ->

$$= P(F \mid E1)P(E1) + P(F \mid E2)P(E2) + \dots + P(F \mid E5)P(E5)$$

–LAW OF TOTAL PROBABILITY–

Conditional Probability - Chain Rule

sometimes we have a **sequential process** and want to find the probability of that

e.g., finding the probability that event E1 happened, then event E2 happens, …. then event En happens

watch out for sometimes when the counting method may be easier

$$P(E1 \cap E2 \cap E3 \cap En) = P(E1) \cdot P(E2 \mid E1) \cdot P(E3 \mid E2 \cap E1) \cdot \ldots \cdot P(En \mid E1 \cap E2 \ldots \cap E(n-1))$$

—CHAIN RULE—

multiplying probability of each event happening conditioned on all the previous events

Connecting Chain rule, Bayes' Rule, & LTP

Chain Rule!

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A)+P(B|A^C)\,P(A^C)}$$

Law of Total Probability!

Independence

Independence

two events are independent if there is no correlation between the events and they don't depend on each other

two events A, B are **statistically** independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

or

$$P(A \mid B) = P(A) \quad \textit{and} \quad P(B \mid A) = P(B)$$

"knowing that B happened doesn't affect the probability that A will happen and vice versa"
"knowing that B happened doesn't give any new information about A"

Just because 2 events may "sound like" they're independent, that doesn't mean that they are *statistically* independent

Conditional Independence

two events A, B are conditionally independent if

$$P((A \cap B) \mid C) = P(A \mid C) \cdot P(B \mid C)$$

Just because 2 events may "sound like" they're independent,
that doesn't mean that they are *statistically* independent

Review Questions?

Question #1

It is always the case that $P(A|B) = P(B|A)$

- True
- False

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- False

Question #2

Suppose A and B are independent events. Then:

- $P(A \cap B) = P(A)P(B)$
- $P(A|B) = P(A)$
- Both are true
- Both are false

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- $P(A|B) = P(A)$
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Question #3

For any two events A and B:

- $P(A|B) = P(A \cap B)$
- $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
- $P(A|B) = \frac{P(B|A)P(B)}{P(A)}$

Question #3

For any two events A and B:

- $P(A|B) = P(A \cap B)$
- $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
- $P(A|B) = \frac{P(B|A)P(B)}{P(A)}$

Question #4

Let A and B be the event that a six-sided die is at most 3 and at least 4, respectively. Then A and B are a partition.

- True
- False

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- True
- False

Question #5

Let T be the event that Alice tests positive for the cold. Let C be the event that Alice actually has the cold. Suppose the probability that Alice tests positive given that she has a cold is 0.8. Then the probability she tests negative given that she has a cold is:

- 0.8
- 0.2
- Not enough information

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- 0.8
- 0.2
- Not enough information

Question #6

For events $A_1, \dots, A_n$ it follows that

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_2 \cap A_1) \cdot \dots \cdot P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

- True
- False

Question #6

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- True
- False

Problem 3 - Balls from an urn

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Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

a) What is the probability that all three balls are all of the same color?

b) What is the probability that we get more than one red ball given the first ball is red?

Work on this problem with the people around you, and then we'll go over it together!

3 - Balls from an Urn

Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

a) What is the probability that all three balls are all of the same color?

The experiment is modeled with $\Omega = \{r, b\}^3$. Probabilities are assigned as we have seen in class, by assuming every draw is uniform among the remaining balls. Then, note that $\mathbb{P}(rrr) = 3/7 \cdot 2/6 \cdot 1/5 = 1/35$ and $\mathbb{P}(bbb) = 4/7 \cdot 3/6 \cdot 2/5 = 4/35$. Therefore, the probability that they all have the same color is $1/35 + 4/35 = 1/7$.

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a) What is the probability that all three balls are all of the same color?

b) What is the probability that we get more than one red ball given the first ball is red?

Let R be the event that the first ball is red. Since we select the first ball uniformly, $\mathbb{P}(R) = \frac{3}{7}$. (This can be computed explicitly from Ω.) We also consider the event M that we have more than one red ball. Let M be the event that more than one ball is red. We need to now compute the probability $\mathbb{P}(M \cap R)$, but note that by the law of total probability

$$\mathbb{P}(M \cap R) = \mathbb{P}(R) - \mathbb{P}(M^c \cap R) = 3/7 - \mathbb{P}(M^c \cap R) .$$

We could compute this probability directly from Ω, but there is an easier way. Note that $M^c \cap R$ is the event that the first ball is red, and both remaining balls are blue. In particular,

$$\mathbb{P}(M^c \cap R) = \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} = \frac{6}{35} .$$

Thus, $\mathbb{P}(M \cap R) = 3/7 - 6/35 = 9/35$, and

$$\mathbb{P}(M|R) = \frac{\mathbb{P}(M \cap R)}{\mathbb{P}(R)} = \frac{9/35}{3/7} = \frac{3}{5} .$$

Problem 5) Marbles in a Pocket

5 – Marbles in a Pocket

Aleks has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If she transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

Work on this problem with the people around you, and then we'll go over it together!

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Let $W\_, B\_$ denote the event that we choose a white marble or a blue marble respectively, with subscripts L, R indicating from which pocket we are picking – left and right, respectively.
We know that we will pick from the left pocket first, and right pocket second. We can then use the Law of Total Probability conditioning on the color of the transferred marble so that:

$$\Pr(B_R) = \Pr(W_L) \cdot \Pr(B_R|W_L) + \Pr(B_L) \cdot \Pr(B_R|B_L) = \frac{3}{8} \cdot \frac{4}{9} + \frac{5}{8} \cdot \frac{5}{9} = \boxed{\frac{37}{72}}$$

Problem 9 – What’s the chance it was raining?

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Suppose that you can commute to campus via bike or public transit. If it's raining, you'll bike to campus 20% of the time, and if it's not raining, you'll bike to campus 60% of the time. You know that it rains 7 days out of 10 during Autumn quarter. If you biked to campus today, what is the probability that it was raining?

Work on this problem with the people around you, and then we'll go over it together!

9 – What's the chance it was raining?

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Let B be the event that you biked to campus, and R be the event that it rained. We want to calculate $P(R|B)$. By Bayes' Rule, we have

$$P(R|B) = \frac{P(B|R) \cdot P(R)}{P(B)}$$

To solve for the denominator, we use the law of total probability:

$$P(B) = P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)$$

So, we have

$$P(R|B) = \frac{P(B|R) \cdot P(R)}{P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)}$$

We are given in the problem the probability of biking given raining or not raining, and the probability of it raining. So, we know the following:

$$P(B|R) = 0.2$$

$$P(B|R^C) = 0.6$$

$$P(R) = 0.7$$

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We know that

$$\begin{aligned} P(R^C) &= 1 - P(R) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

Plugging in these values, we get that

$$\begin{aligned} P(R|B) &= \frac{P(B|R) \cdot P(R)}{P(R) \cdot P(B|R) + P(R^C) \cdot P(B|R^C)} \\ &= \frac{0.2 \cdot 0.7}{0.2 \cdot 0.7 + 0.6 \cdot 0.3} \\ &= \frac{0.14}{0.14 + 0.18} \\ &= 0.4375 \end{aligned}$$

Problem 10 – A Game

10 – A Game

Howard and Jerome are playing the following game: A 6-sided die is thrown and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers.

- If it shows 5, Howard wins.
- If it shows 1, 2, or 6, Jerome wins.
- Otherwise, they play a second round and so on.

What is the probability that Jerome wins on the 4th round?

Work on this problem with the people around you, and then we'll go over it together!

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Let J_i be the event that Jerome wins on the i-th round, and let N_i be the event that nobody wins on the i-th round. Then we are interested in the event:

$$N_1 \cap N_2 \cap N_3 \cap J_4$$

Using the chain rule, we have:

$$\begin{aligned} \mathbb{P}(N_1, N_2, N_3, J_4) &= \mathbb{P}(N_1) \cdot \mathbb{P}(N_2 \mid N_1) \cdot \mathbb{P}(N_3 \mid N_1, N_2) \cdot \mathbb{P}(J_4 \mid N_1, N_2, N_3) \\ &= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{54}} \end{aligned}$$

In the final step, we used the fact that if the game hasn't ended, then the probability that it continues for another round is the probability that the die comes up 3 or 4, which has probability $2/6 = 1/3$. The probability Jerome wins a given round is the probability it lands on 1, 2, or 6, which is $3/6 = 1/2$.

Problem 11 – Another Game

11 - Another Game

Alice and Alicia are playing a tournament in which they stop as soon as one of them wins n games. Alicia wins each game with probability p and Alice wins with probability $1 - p$, independently of other games. What is the probability that Alicia wins and that when the match is over, Alice has won k games?

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Since the match is over when someone wins their n^{th} game, and Alicia won the overall match, Alicia must have won the very last game.
Before this final game, Alicia must have won exactly $n-1$ games, and Alice must have won exactly k games.

Therefore, the probability that we reach a point in time when Alicia has won $n-1$ games and Alice has won k games is:

$$p^{n-1} \cdot (1-p)^k \cdot \binom{n-1+k}{k}$$

The binomial coefficient counts the number of ways of distributing the k games that Alice won among the $n-1+k$ total games played so far.

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At that point, in the event of interest, Alicia wins the next game. This happens with probability p, independent of previous outcomes. Therefore, our final probability is:

$$\left[p^{n-1} \cdot (1-p)^k \cdot \binom{n-1+k}{k}\right] \cdot p = \boxed{p^n \cdot (1-p)^k \cdot \binom{n-1+k}{k}}$$

That's All, Folks!

Thanks for coming to section this week!
Any questions?