CSE 312 Section 2

Intro to Discrete Probability

Administrivia

Announcements & Reminders

- Office Hours
  - Offered on Mon, Tue, Wed, and Fri! Mostly in-person, a few zoom options
  - Times posted on course website

- Pset1
  - Due yesterday, Wednesday 7/1 @ 11:59pm
  - Late deadline Friday 7/3 @ 11:59pm (max of 2 late days per homework)

- Pset2
  - Released on the course website
  - Due Wednesday 7/8@ 11:59pm

- No class friday

Homework

- Submissions
  - LaTeX (highly encouraged)
    - overleaf.com
    - Many TAs are familiar with Overleaf and using LaTeX
  - Word Editor that supports mathematical equations
  - Handwritten neatly and scanned
- Homework will typically be due on Wednesdays at 11:59pm on Gradescope
- Each assignment can be submitted a max of 72 hours late
- You have **6 late days total** to use throughout the quarter
  - Anything beyond that will result in a deduction on further late assignments

Feeling Stuck?

- Many of the strategies you need to solve homework problems are usually **practiced in section**
- For Pset 2, review Problems 4, 5, 6 on the section handout today to practice Pigeonhole Principle.
- Review Problem 8 to practice Stars and Bars.
- Review Problem 2 to practice combinatorial proofs.
- There are many ways to get help! EdStem, office hours, lecture slides, etc.

Any lingering questions from this last week?

Review

Binomial Theorem

Statement:

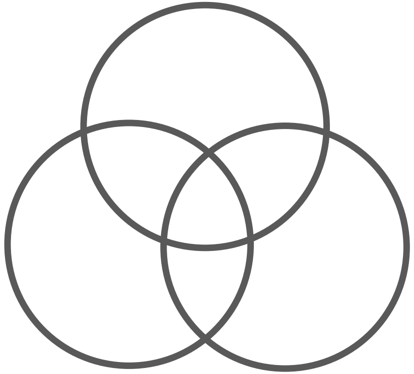
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Inclusion-Exclusion

$|A \cup B| = |A|+|B|-|A \cap B|$



$|A \cup B \cup C| =$

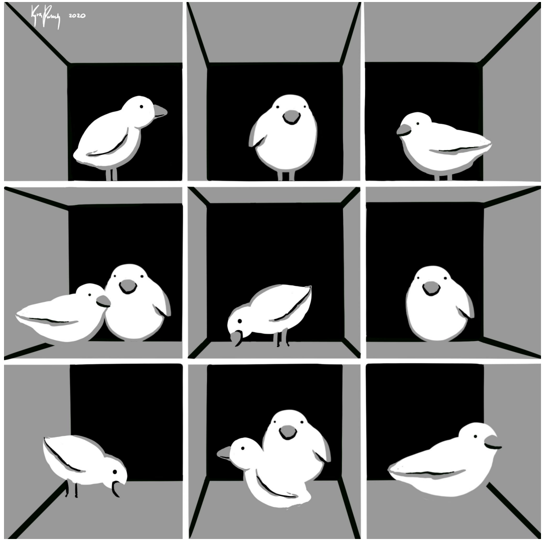
singles - doubles + triples

$|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$

Pigeonhole Principle

If there are **n** pigeons with not enough holes for them to stay in (**k** to be exact), what can we say about at least how many pigeons at least one hole will hold?

ceil(n / k)



Complementary Counting

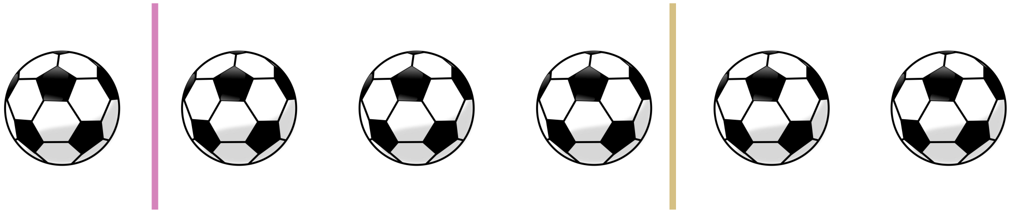
If asked to find the number of ways to do *X*, you can:
(1) find the total number of ways to do everything
(2) subtract the number of ways to <u>*not*</u> do *X*.

Stars and Bars

How many ways can you distribute **n** indistinguishable balls into **k** distinguishable bins?

*Arrange **n** balls and **k - 1** dividers*

$${}_{(n + k - 1)}C_{n}$$



Probability!

- **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S
- **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
- **Union:** The union of two events E and F is denoted $E \cup F$
- **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
- **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$
- **Complement:** The complement of an event E is denoted E^C or $\bar{E}$ or $\neg E$, and is equal to $\Omega \backslash \mathrm{E}$
- **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$

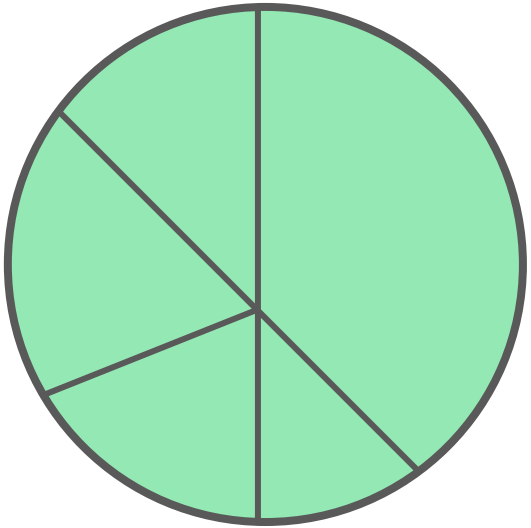
Probability!

- **Axioms of Probability**
  - **Non-negativity:** For any event E, $\mathbb{P}(E) \geq 0$
  - **Normalization:** $\mathbb{P}(\Omega) = 1$
  - **Additivity:** If E and F are mutually exclusive events, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

- **Corollaries of these axioms**
  - **Complementation**: $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$
  - **Monotonicity**: If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$
  - **Inclusion-Exclusion**: $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

- **Equally Likely Outcomes**: If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$

Probability!



Sample Space

An event is a subset of the sample space

$E \subseteq \Omega$

If each outcome in the sample space is *equally likely*, the probability of an event is

`P(E) = |E| / |Ω|`

If the union of a set of mutually exclusive events is equal to the sample sets, those events *partition* the sample space

Some Practice Problems

Problem 1, (d, e, f, g)

1d)

An **event** and **sample space** are, respectively:

- ○ The total set of possible outcomes; A subset of the event space
- ○ A subset of the sample space; The total set of possible outcomes
- ○ Some set of outcomes; Any other set of outcomes.

1d)

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- ○ The total set of possible outcomes; A subset of the event space
- ○ A subset of the sample space; The total set of possible outcomes
- ○ Some set of outcomes; Any other set of outcomes.

(b). By definition, the sample space (Ω) encompasses all possible outcomes, and an event (E) is always a subset of the sample space ($E \subseteq \Omega$).

1e)

True or **False**. It is always true that $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

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True or **False**. It is always true that $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

False. This formula only holds if all outcomes in the sample space Ω are *equally likely*.

1f)

If A is the event that I eat an apple today,

- ○ $\overline{A}$ is the event that I eat a banana today, and $P(A) + P(\overline{A}) = 0.5$
- ○ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 0$
- ○ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 1$

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- ○ $\overline{A}$ is the event that I eat a banana today, and $P(A) + P(\overline{A}) = 0.5$
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- ○ $\overline{A}$ is the event that I do not eat an apple today, and $P(A) + P(\overline{A}) = 1$

(c). $\overline{A}$ is the *complement* of event A (the event that A does *not* happen). Because the sample space probabilities must normalize to 1, the probability of an event plus its complement is always exactly 1.

1g)

True or **False**. For any two events A and B, $P(A \cup B) > P(A) + P(B)$.

1g)

True or **False**. For any two events A and B, $P(A \cup B) > P(A) + P(B)$.

False. The correct inequality is $P(A \cup B) \leq P(A) + P(B)$. By the property of inclusion-exclusion for probability, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Because probabilities must be non-negative, subtracting $P(A \cap B)$ means the union can never be strictly greater than the sum of the individual probabilities.

Problem 2, (b) - Thinking Combinatorially

2b) – Thinking Combinatorially

Give a combinatorial proof of

$$\binom{a+b}{a} = \sum_{i=0}^{a} \binom{a}{i} \binom{b}{i}.$$

(Note that $\binom{a}{b}$ is 0 if $b > a$.)

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Hint: count the number of ways to choose a team of a people from a larger pool consisting of a right-handed people and b left-handed people (in total of $a + b$ people)

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Hint: count the number of ways to choose a team of a people from a larger pool consisting of a right-handed people and b left-handed people (in total of $a + b$ people)

(also remember to state the conclusion explicitly!)

2b) – Thinking Combinatorially

Give a combinatorial proof of

$$\binom{a+b}{a} = \sum_{i=0}^{a} \binom{a}{i}\binom{b}{i}.$$

(Note that $\binom{a}{b}$ is 0 if $b > a$.)

Scenario: We want to count the number of ways to choose a team of a people from a larger pool consisting of a right-handed people and b left-handed people (for a total of $a + b$ people).

- **Left Hand Side $\binom{a+b}{a}$:** Choose the team of a people directly from the entire pool of $a + b$ people. There are $\binom{a+b}{a}$ ways to do this.
- **Right Hand Side $\sum_{i=0}^{a} \binom{a}{i}\binom{b}{i}$:** We partition the team selection based on the number of left-handed people on the team, which we will call i. The value of i can range from 0 to a.
  - **Step 1:** Choose i left-handed people from the pool of b left-handed people ($\binom{b}{i}$ ways).
  - **Step 2:** To complete the team of a people, we must choose $a - i$ right-handed people from the pool of a right-handed people ($\binom{a}{a-i}$ ways). Since choosing $a - i$ people to *be* on the team is identical to choosing i people to *exclude* from the team, $\binom{a}{a-i} = \binom{a}{i}$.

  Applying the Product Rule, for a specific i, there are $\binom{a}{i}\binom{b}{i}$ ways to form the team. By the Sum Rule, we sum across all possible values of i to get $\sum_{i=0}^{a} \binom{a}{i}\binom{b}{i}$.

Conclusion: Since both sides count the exact same scenario, they are equal.

Problem 8 – Count the Solutions

8 – Count the Solutions

Consider the following equation: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the equation? (Hint: Use the "Stars and Bars" method.)

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- **Stars (n):** Think of the sum total 70 as a sequence of 70 indistinguishable units (stars).
- **Bars ($k-1$):** We want to split these 70 units into $k = 6$ distinct variables (buckets). To split a line into 6 segments, we need to place 5 dividers (bars) among the stars.

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We use the "Stars and Bars" method (combinations with repetition).

- **Stars (n):** Think of the sum total 70 as a sequence of 70 indistinguishable units (stars).
- **Bars ($k-1$):** We want to split these 70 units into $k=6$ distinct variables (buckets). To split a line into 6 segments, we need to place 5 dividers (bars) among the stars.

Because the variables can be nonnegative (including 0), blocks are allowed to be empty, which means bars can be placed adjacent to each other. We are simply choosing the positions for the 5 bars out of the total $70+5$ available slots.

$$\binom{70+6-1}{6-1} = \binom{75}{5} = \boxed{17{,}259{,}390}$$

8 – Count the Solutions

Follow Up:

How many different solutions are there if each a_i must be at least 2?

Work on this problem with the people around you, and then we'll go over it together!

Hint: How do we apply the restriction to change the total first?
Think about counting solutions b_i, where $b_i = a_i + 2$

8 – Count the Solutions

Follow Up:

How many different solutions are there if each a_i must be at least 2?

We use the Stars and Bars method, but must handle the restriction first.

- **Step 1: Satisfy the restriction.** Let $a_i' = a_i + 2$. Them it must be that $a_1' + a_2' + a_3' + a_4' + a_5' + a_6' = 70 - 2 \cdot 6 = 58$
- **Step 2: Solve the remaining problem as before**
  - **Stars (n):** The 58 units that we have left to partition among the a_i' variables
  - **Bars ($k-1$):** To divide into $k = 6$ segments, we need $6 - 1 = 5$ bars.

Thus, we must choose the position of the 5 bars out of the total of $58 + 5$ positions. This yields $\boxed{\binom{63}{5}}$ ways. (Equivalently, $\binom{63}{58}$).

Problem 4 – Pigeonhole Practice

4) – Pigeonhole Practice

Either give the minimum number of people that must be in a group to ensure that the property holds, or else indicate that the property need not hold even for arbitrarily large groups of people.

4a). At least 2 people were born on the same day of the year (ignore year of birth)

4) – Pigeonhole Practice

Either give the minimum number of people that must be in a group to ensure that the property holds, or else indicate that the property need not hold even for arbitrarily large groups of people.

4a). At least 2 people were born on the same day of the year (ignore year of birth)

- **Pigeons:** The n people in the group.
- **Holes:** The 365 possible days of the year.
- **Mapping:** Each person is mapped to the day of the year on which they were born.

4) – Pigeonhole Practice

4a). At least 2 people were born on the same day of the year (ignore year of birth)

- **Pigeons:** The n people in the group.
- **Holes:** The 365 possible days of the year.
- **Mapping:** Each person is mapped to the day of the year on which they were born.

$$\left\lceil \frac{n}{365} \right\rceil \geq 2. \quad \Longrightarrow \quad \boxed{366}$$

4) – Pigeonhole Practice

Either give the minimum number of people that must be in a group to ensure that the property holds, or else indicate that the property need not hold even for arbitrarily large groups of people.

4b). At least 2 people were born on April 1.

- **Pigeons:** ???
- **Holes:** ???
- **Mapping:** ???

4) – Pigeonhole Practice

4b). At least 2 people were born on April 1.

The property need not hold, even for arbitrarily large groups.

The Pigeonhole Principle guarantees that some day of the year must be shared once the group is large enough, but it does not guarantee that a specific day is shared.

For example, it is possible to have an arbitrarily large group of people in which everyone was born on April 2. In that case, no one was born on April 1, so the property does not hold

Problem 10 – Spades and Hearts

10) – Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them uniformly at random. What is the sample space and how big it is (i.e., what what is its cardinality)? Compute P(E), where E is the event that the suits of the shuffled car ds are in alternating order.

Hint: There are two disjoint cases for the starting suit.

Problem 11 – Shuffling Cards

11 – Shuffling Cards

We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace > King > Queen > Jack > $10 > \cdots > 2$. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely). What is the probability the first card on the deck is (strictly) larger than the second one?

Work on this problem with the people around you, and then we'll go over it together!

Hint: Only need to consider the sample space for the first two cards.

That’s All, Folks!

Thanks for coming to section this week!
Any questions?