Joint Distributions

CSE 312 26Su

Lecture 17

Announcements

Homework 5 due Wednesday

Quiz 5 on Friday

Where Are We?

Last time:

- Polling

Today:

- Joint Distributions
  - Joint Support
  - Joint PMF / PDF
  - Joint CDF
  - Joint Expectation
- Conditional Distributions
  - Conditional Expectation
- Law of Total Expectation
- Covariance

Multiple Random Variables

Analyzing *Multiple* Random Variables

So far, we've pretty much only analyzed one random variable at a time

We've *worked* with multiple random variables (e.g., the sum of them) but haven't really analyzed their relationships except for independence

Today, we will talk about <u>analyzing the relations between multiple RVs</u>

Analyzing *Multiple* Random Variables

Examples:

- **Economics**: Analyzing the relationship between stock returns and volume helps in understanding market behavior and making investment decisions
- **Healthcare:** Understanding how different factors (like blood pressure and cholesterol levels) affects the probability of medical conditionals
- **Machine learning**: understanding how different features contribute to predicting a label and improving model accuracy (what features are most important)
- **Recommendation systems**: Understanding relationship between user preferences and item features to improve recommendation systems, or other factors like age and choices of products

Joint Distributions

We'll start with the **discrete** case

We have two **discrete** random variables X and Y
(that may or may not be independent)

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Joint **Support/Range** - $\Omega_{X,Y}$

$\Omega_{X,Y}$ is the set of all possible **pairs** of values X and Y can be together

> $\Omega_{X,Y} = \{(a, b) : p_{X,Y}(a, b) > 0\} \subseteq \Omega_X \times \Omega_Y$

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Joint **PMF** *(probability mass function)*- $\mathrm{p}_{X,Y}(a, b)$

$\mathrm{p}_{X,Y}(a, b)$ defines probabilities of X and Y being certain values at once

> $p_{X,Y}(a, b) = \mathbb{P}(X = a \cap Y = b) = \mathbb{P}(X = a, Y = b)$

*should be defined for **all** values of a and b*

Normalization Property: $\sum_{(a,b)\in\Omega_{X,Y}} p_{X,Y}(a, b) = 1$

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> *should be defined for* ***all*** *values of a and b*

Joint **CDF** *(cumulative distribution function)* - $\mathrm{F}_{X,Y}(a, b)$

> $F_{X,Y}(a, b) = \mathbb{P}(X \le a \cap Y \le b) = \mathbb{P}(X \le a, Y \le b)$
> *should be defined for* ***all*** *values of a and b*

We have two **discrete** random variables X and Y
(that may or may not be independent)

(Joint) **Independence**

X and Y are independent if:

> $p_{X,Y}(a,b) = p_X(a) \cdot p_Y(b)$ for all $(a,b) \in \Omega_{X,Y}$

> $\Omega_{X,Y} = \Omega_X \times \Omega_Y$
> *if you find that $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$, that is enough to show that they aren't independent*

This is the same definition we've seen before for independence of random variables
$\mathbb{P}(X = a, Y = b) = \mathbb{P}(X = a) \cdot \mathbb{P}(X = b)$

08/03: missing symbol corrected

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let X be the value of the 1st dice
Let Y be the value of the 2nd dice

$\Omega_X = \Omega_Y = \{1,2,3,4\}$

The **joint support** is $\Omega_{X,Y} = \Omega_X \times \Omega_Y$
because all combinations are possible

What is the **joint PMF?**

$p_{X,Y}(a, b) =$

$p_{X,Y}$	Y=1	Y=2	Y=3	Y=4
X=1				
X=2				
X=3				
X=4				

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What is the **joint PMF**?

$$p_{X,Y}(a,b) = \begin{cases} 1/16 & \text{if } a,b \in \Omega_{X,Y} \\ 0 & \text{otherwise} \end{cases}$$

$p_{X,Y}$	Y=1	Y=2	Y=3	Y=4
X=1	1/16	1/16	1/16	1/16
X=2	1/16	1/16	1/16	1/16
X=3	1/16	1/16	1/16	1/16
X=4	1/16	1/16	1/16	1/16

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$
Let W be the max value $W = \max(X, Y)$

$\Omega_U = \Omega_W = \{1,2,3,4\}$

The **joint support** is:

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1				
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The **joint support** is:
$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W; u \leq w\}$

$p_{U,W}$	W=1	W=2	W=3	W=4
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U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

What is the **joint PMF?**

$$p_{U,W}(u, w) = \begin{cases} 1/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u = w \\ 2/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u < w \\ 0 & \text{otherwise} \end{cases}$$

Example: Weird Dice

Roll 2 fair 4-sided dice independently

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What is the **joint PMF?**

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U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

$$p_{U,W}(u, w) = \begin{cases} 1/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u = w \\ 2/16 & \text{if } (u, w) \in \Omega_{U,W} \text{ and } u < w \\ 0 & \text{otherwise} \end{cases}$$

U and W are **NOT independent** because $\Omega_{U,W} \neq \Omega_U \times \Omega_W$

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$. Let W be the max value $W = \max(X, Y)$

Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ *directly.*
Can we find it from the joint PMF $p_{U,W}(u, w)$*?*

We know $\mathbb{P}(U = 1 \cap W = w)$ for all w…

$\mathbb{P}(U = 1)$
$=$

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
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Can we find it from the joint PMF $p_{U,W}(u, w)$?

We know $\mathbb{P}(U = 1 \cap W = w)$ for all w...

$$\mathbb{P}(U = 1)$$
$$= \mathbb{P}(U = 1 \cap W = 1) + \mathbb{P}(U = 1 \cap W = 2) + \mathbb{P}(U = 1 \cap W = 3) + \mathbb{P}(U = 1 \cap W = 4)$$
$$= \frac{1}{16} + \frac{2}{16} + \frac{2}{16} + \frac{2}{16} = \frac{7}{16}$$

Use LoTP because the events $W = 1$, $W = 2$, $W = 3$, $W = 4$ ***partition the sample space***

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

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Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ *directly.*
Can we find it from the joint PMF $p_{U,W}(u, w)$*?*

In general...

$$\begin{aligned}
&\mathbb{P}(U = u) \\
&= \mathbb{P}(U = u \cap W = 1) + \mathbb{P}(U = u \cap W = 2) \\
&\quad + \mathbb{P}(U = u \cap W = 3) + \mathbb{P}(U = u \cap W = 4) \\
&= \sum_{w \in \Omega_W} p_{U,W}(u, w)
\end{aligned}$$

Use ***LoTP*** *because the events* $W = 1$, $W = 2$, $W = 3$, $W = 4$ ***partition the sample space***

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
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Example: Weird Dice

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Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ *directly.*
Can we find it from the joint PMF $p_{U,W}(u, w)$*?*

In general…

$\mathbb{P}(U = u) = \sum_{w \in \Omega_W} p_{U,W}(u, w)$

Use LoTP because the events $W = 1$, $W = 2$, $W = 3$, $W = 4$ ***partition the sample space***

$p_U(u)$ is the "marginal" PMF for U
(because we "marginalized" W)

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

08/03: V typo corrected to W

Marginal Distribution

If you have the joint PMF for two random variables, you can find the PMF of one of them by **using the law of total probability**, **partitioning on the values of the other random variable:**

$$\boxed{p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a,b)}$$

^ it's the same PMF we've talked about before, the "marginal" is just there to indicate is was derived from a joint PMF

Example: Weird Dice

Roll 2 fair 4-sided dice independently

Let U be the min value $U = \min(X, Y)$. Let W be the max value $W = \max(X, Y)$

Suppose I didn't know how to compute $p_U(u) = \mathbb{P}(U = u)$ *directly.*
Can we find it from the joint PMF $p_{U,W}(u, w)$*?*

$$p_U(u) = \begin{cases} \frac{7}{16} & \text{if } u = 1 \\ \frac{5}{16} & \text{if } u = 2 \\ \frac{3}{16} & \text{if } u = 3 \\ \frac{1}{16} & \text{if } u = 4 \\ 0 & \text{otherwise} \end{cases}$$

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Joint **Expectation**

Joint Expectations

For a function $g(X, Y)$, the expectation can be written in terms of the joint PMF.

$$\mathbb{E}[\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y})] = \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{\Omega}_{\boldsymbol{X},\mathbf{Y}}} \boldsymbol{g}(\boldsymbol{a}, \boldsymbol{b}) \cdot \boldsymbol{p}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{a}, \boldsymbol{b})$$

Same ideas as before!

Examples of joint functions:

$g(X, Y) = X + Y$, $g(X, Y) = XY$, $g(X, Y) = X^Y$, $g(X, Y) = \binom{X}{Y}$, *etc.*

Expectation of a function of two RVs

What's $\mathbb{E}[UW]$ for U, W from the last slide?

08/03: typo corrected

Expectation of a function of two RVs

What's $\mathbb{E}[UW]$ for U, W from the last slide?

$$\sum_{u \in \Omega_U} \sum_{w \in \Omega_W} uw \cdot p_{U,W}(u, w)$$

$$= 1 \cdot 1 \cdot \frac{1}{16} + 1 \cdot 2 \cdot \frac{2}{16} + 1 \cdot 3 \cdot \frac{2}{16} + 1 \cdot 4 \cdot \frac{2}{16} + \ 2 \cdot 2 \cdot \frac{1}{16} + 2 \cdot 3 \cdot \frac{2}{16} + 2 \cdot 4 \cdot \frac{2}{16} + 3 \cdot 3 \cdot \frac{1}{16} + 3 \cdot 4 \cdot \frac{2}{16} + 4 \cdot 4 \cdot \frac{1}{16}$$

$$= \frac{100}{16} = \frac{25}{4} = 6.25$$

08/03: typo corrected, missing term added, calculation corrected

Conditional Expectation

Waaaaaay back when, we said conditioning on an event creates a new probability space, with all the laws holding.

So we can define things like "conditional expectations" which is the expectation of a random variable in that new probability space.

$$\mathbb{E}[X|E] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x|E)$$

$$\mathbb{E}[X|Y = y] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x|Y = y)$$

Conditional Expectations

All your favorite theorems are still true.

For example, linearity of expectation still holds

$$\mathbb{E}[(aX + bY + c)\ |E] = a\mathbb{E}[X|E] + b\mathbb{E}[Y|E] + c$$

Law of Total Expectation

Let $A_1, A_2, \ldots, A_k$ be a partition of the sample space, then

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i]\mathbb{P}(A_i)$$

Let X, Y be discrete random variables, then

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X|Y = y]\mathbb{P}(Y = y)$$

Similar in form to law of total probability, and the proof goes that way as well.

LTE

You will flip 2 (independent, fair coins). Call the number of heads X. Then (independently of the coin flips) draw an exponential random variable Y from the distribution $\text{Exp}(X+1)$.

What is $\mathbb{E}[Y]$?

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What is $\mathbb{E}[Y]$?

$\mathbb{E}[Y]$

$= \mathbb{E}[Y|X=0]\mathbb{P}(X=0) + \mathbb{E}[Y|X=1]\mathbb{P}(X=1) + \mathbb{E}[Y|X=2]\mathbb{P}(X=2)$

$= \mathbb{E}[Y|X=0] \cdot \frac{1}{4} + \mathbb{E}[Y|X=1] \cdot \frac{1}{2} + \mathbb{E}[Y|X=2] \cdot \frac{1}{4}$

$= \frac{1}{0+1} \cdot \frac{1}{4} + \frac{1}{1+1} \cdot \frac{1}{2} + \frac{1}{2+1} \cdot \frac{1}{4} = \frac{7}{12}$

Different dice

Roll two fair dice independently. Let U be the minimum of the two rolls and W be the maximum

What is $\mathbb{P}(U = 2|W = 3)$?

$$\frac{\mathbb{P}(U=2 \cap W=3)}{\mathbb{P}(W=3)} = \frac{2/16}{5/16} = \frac{2}{5}$$

$$p_{U|W}(2|3) = \frac{2}{5}$$

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

08/03: typo corrected

Different dice

Find these values

$p_{W|U}(2|1) =$

$p_{U|W}(1|2) =$

$p_{U|W}(4|1) =$

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Different dice

Find these values

$$p_{W|U}(2|1) = \frac{p_{W,U}(2,1)}{p_U(1)} = \frac{2/16}{7/16} = \frac{2}{7}$$

$$p_{U|W}(1|2) = \frac{p_{U,W}(1,2)}{p_W(2)} = \frac{2/16}{3/16} = \frac{2}{3}$$

$$p_{U|W}(4|1) = \frac{p_{U,W}(4,1)}{p_W(1)} = \frac{0}{1/16} = 0$$

$p_{U,W}$	W=1	W=2	W=3	W=4
U=1	1/16	2/16	2/16	2/16
U=2	0	1/16	2/16	2/16
U=3	0	0	1/16	2/16
U=4	0	0	0	1/16

Joint Distributions

For **continuous** joint distributions, everything will look very similar but we *replace summations with integrals*, and use a *density function instead of the PMF*

We have two **continuous** random variables X and Y
(that may or may not be independent)

Joint **Support/Range** - $\Omega_{X,Y}$

$\Omega_{X,Y} = \{(a, b) : f_{X,Y}(a, b) > 0\} \subseteq \Omega_X \times \Omega_Y$

Joint **PDF** - $f_{X,Y}(a, b)$

$f_{X,Y}(a, b)$ *defined for **all*** $(a, b) \in \mathbb{R} \times \mathbb{R}$

Normalization Property:

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx\, dy = 1$

Joint **CDF** - $F_{X,Y}(a, b)$

$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b)$
*defined for **all*** $(a, b) \in \mathbb{R} \times \mathbb{R}$

Joint **Independence**

> $f_{X,Y}(a, b) = f_X(a) \cdot f_X(b)$ for all $(a, b) \in \Omega_{X,Y}$
> $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Joint **Expectation**

$\mathbb{E}[g(X, Y)] =$
$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, f_{X,Y}(x, y) dx\, dy$

Marginal **PDF** - $f_X(x)$, $f_Y(y)$

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$

Notice we're integrating (summing) over what the other RV can be

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the origin. X and Y are the x and y coordinates of the point the dart lands at.

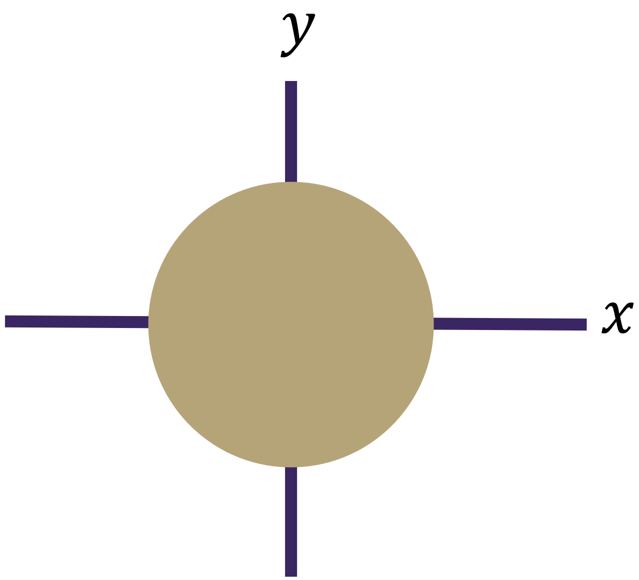
Let's find and sketch the **joint range** $\Omega_{X,Y}$

08/03: 'version' typo corrected to 'origin'

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

Let's find and sketch the **joint range** $\Omega_{X,Y}$



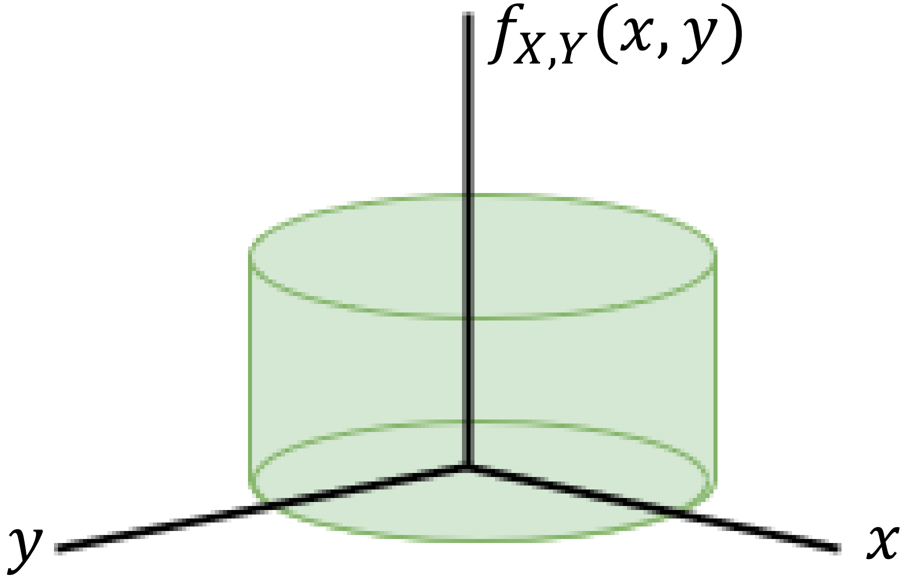
$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R} \times \mathbb{R}; \mathrm{x}^2 + \mathrm{y}^2 \leq r^2\}$$

08/03: circle equation typo corrected to r^2

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

Let's find and sketch the **joint PDF** $f_{X,Y}(x,y)$



$$f_{X,Y}(x,y) = \begin{cases} & \\ & \end{cases}$$

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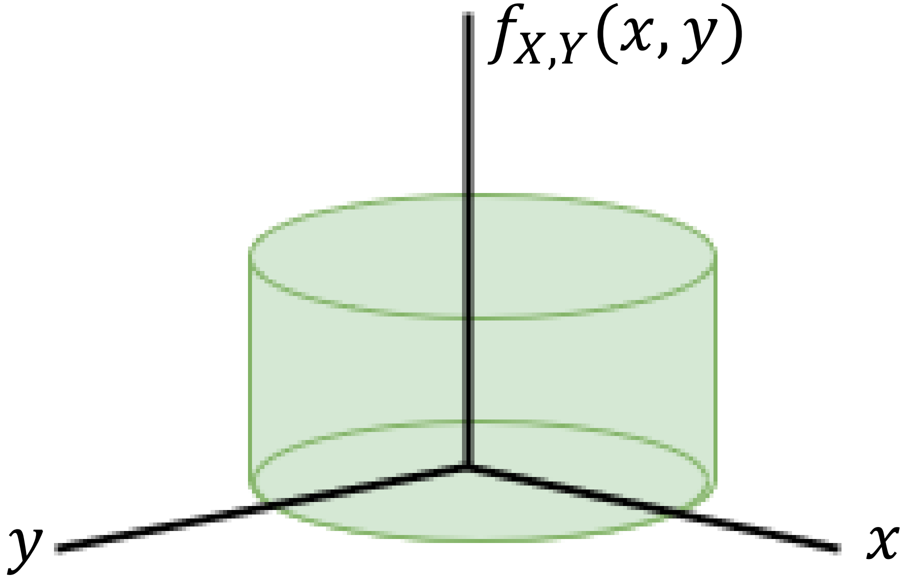

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi r^2} & x^2 + y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

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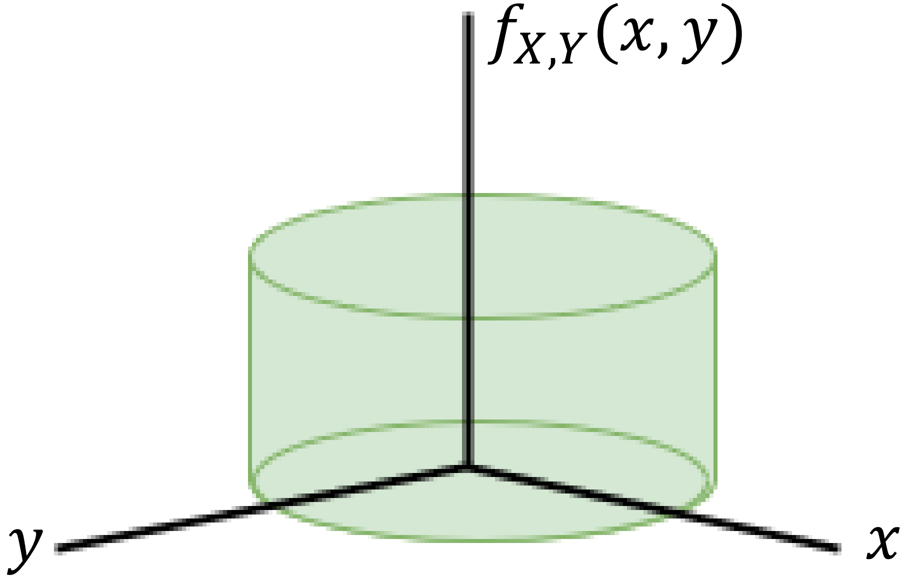
What is the marginal PDF for X and Y?



$f_X(a) =$

$f_Y(b) =$

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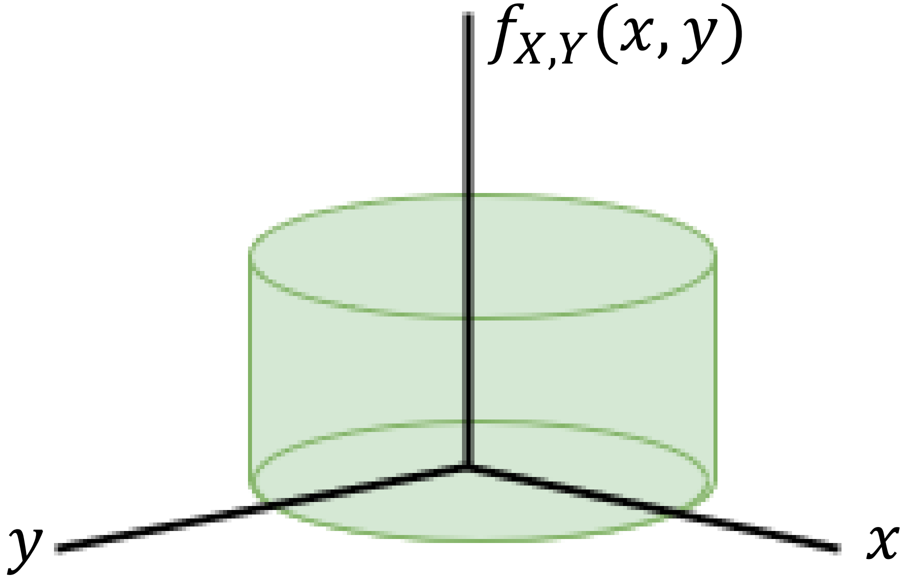


$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy$$

$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)dx$$

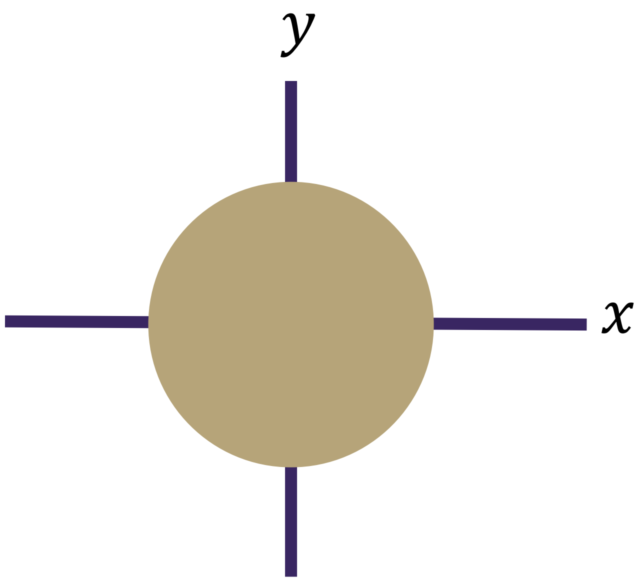
08/03: circle equation typo corrected to r^2

Example: Darts

We throw a dart uniformly at random onto a circle of radius r centered around the version. X and Y are the x and y coordinates of the point the dart lands at.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

What is the marginal PDF for X and Y?



$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy$$

$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)dx$$

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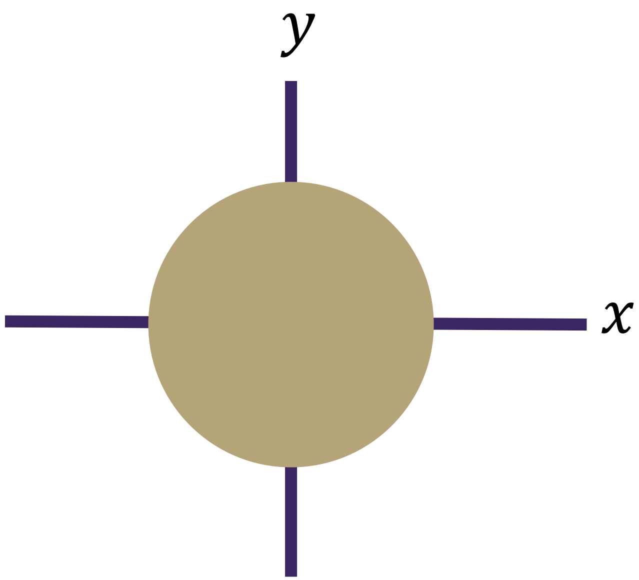

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy = \int_{-\sqrt{r^2-a^2}}^{\sqrt{r^2-a^2}} \frac{1}{\pi r^2} dy = \frac{2\sqrt{r^2 - a^2}}{\pi r^2}$$

$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)dx = \int_{-\sqrt{r^2-b^2}}^{\sqrt{r^2-b^2}} \frac{1}{\pi r^2} dx = \frac{2\sqrt{r^2 - b^2}}{\pi r^2}$$

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Joint Continuous Probabilities

Just like we've done with PDFs for single random variables…

$$\mathbb{P}(a \leq X \leq b \cap c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y) dy\, dx$$

Analogues for continuous

Everything we saw today has a continuous version.

There are "no surprises"– replace pmf with pdf and sums with integrals.

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X = x, Y = y)$	$f_{X,Y}(x,y) \neq P(X = x, Y = y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Conditional PMF/PDF	$p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X \mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$

A note on independence

The definition of independence says X, Y independent if and only if

$p_{X,Y}(x, y) = p_X(x)p_Y(y)$ or $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ (as appropriate)

There's often a nice shortcut. If X, Y are independent then joint support of X, Y (denoted $\Omega_{X,Y}$) must be $\Omega_X \times \Omega_Y$.

Joint support is $\{(x, y): p_{X,Y}(x, y) > 0\}$.

Often easier to verify <u>dependence</u> when those are different (especially in the continuous case).

But note this is a single implication not an if-and-only-if.